# Visualizing a nowhere differentiable but continuous everywhere function

Wei-Chi Yang
Radford University
Virginia, VA 24142
www.radford.edu/wyang

## Objective

In advanced calculus, we learned that if

$$f(x) = \sum_{k=1}^{\infty} a^k \cos b^k \pi x$$

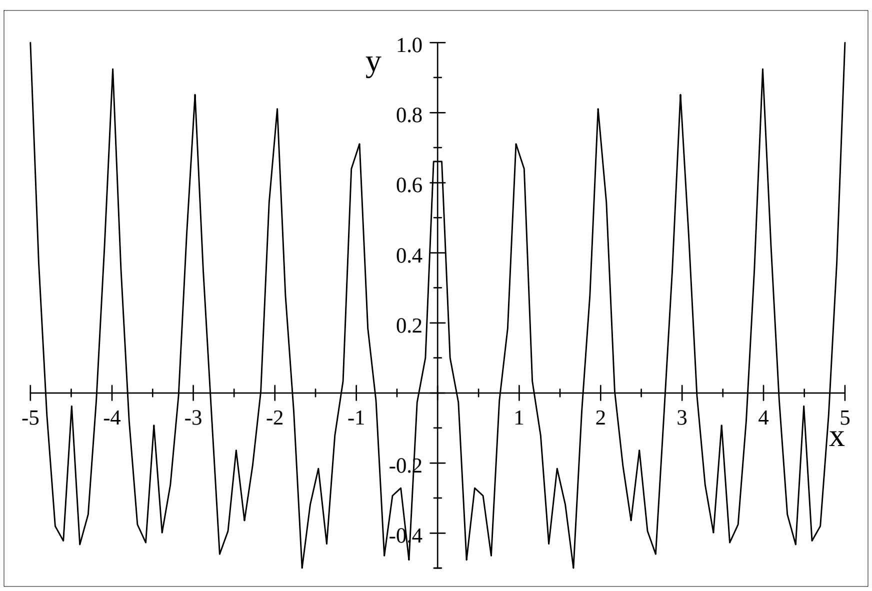
where $a$ and $b$ satisfy certain relationship ($0 < a < 1, b \in Z^+$ and $ab > 1 + \frac{3}{2}\pi = 5.712388981$), then we can prove that the function is nowhere differentiable but continuous everywhere. In this note we shall use the graphing approaches to discover how the behavior of $a/b$ will lead us to a desired nowhere differentiable but continuous function. For a detailed construction, see [1]. We know we can't plot an infinite series of functions but we certainly can use the partial sum to predict the graph of an infinite sum.

## Experimenting the graphs of partial sums

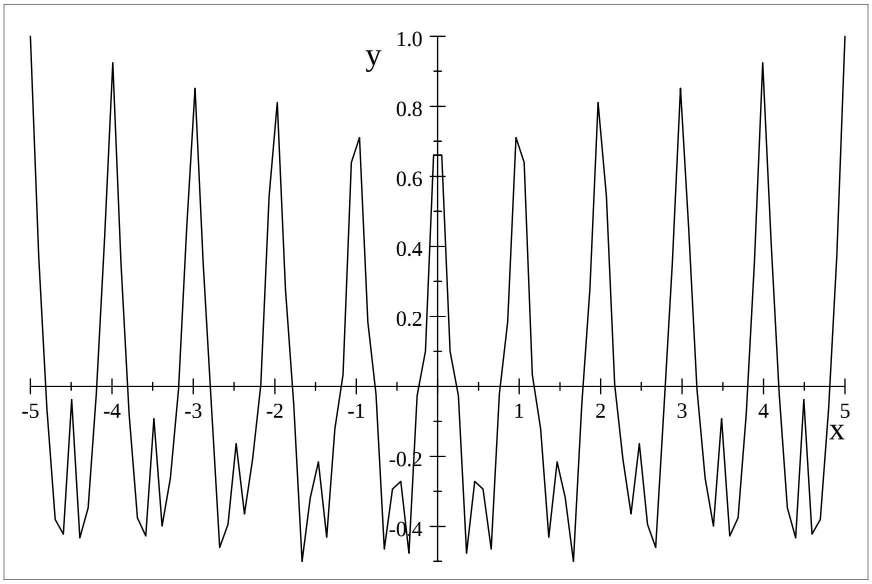
First we define the partial sum function as follows.

$$F(a,b,x,n) = \sum_{k=1}^{n} a^k \cos b^k \pi x$$

By setting $a = 1/2, b = 2,$ and $n = 20$, we graph the function
$F(1/2,2,x,20)$




Let's increase the partial sum from $n = 20$ to $n = 30$, and we obtain the following graph.
$F(1/2,2,x,30)$




The graph of $F(1/2, 2, x, 30)$ seems to be similar to that of $F(1/2, 2, x, 20)$. We could zoom in many times to obtain the graph of $F(1/2, 2, x, 30)$ again as follows:

$F(1/2, 2, x, 30)$

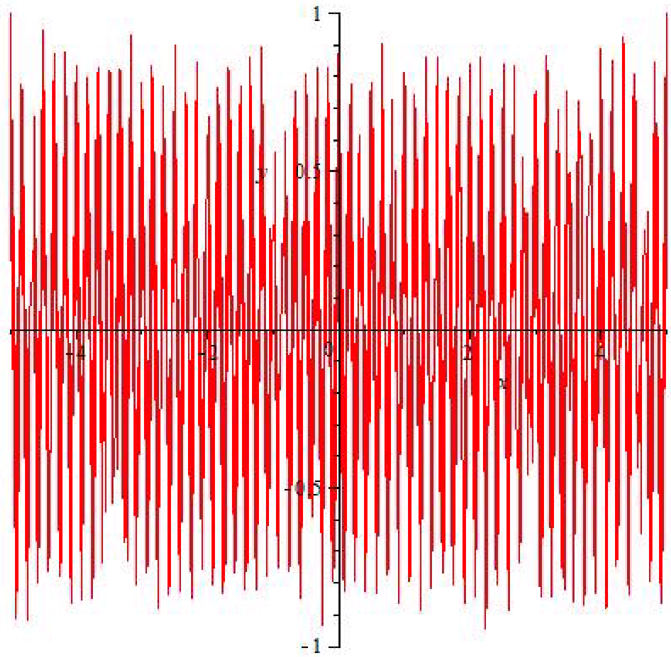
Notice that for $a = 1/2$, and $b = 2$, even if we increase the partial sum, we don't have a function that oscillates as much as we want yet. Therefore, we consider to increase the ratio of $\frac{b}{a}$ from 4 to 8 as follows:

$F(1/2, 4, x, 30)$

Note that it oscillates more than the previous graph, but still not as much as what we like. We zoom in the graph of $F(1/2, 4, x, 30)$ as follows:

$F(1/2, 4, x, 30)$

We see that the function, $F(1/2, 4, x, 30)$ does have more spikes than those of $F(1/2, 2, x, 30)$ and $F(1/2, 4, x, 30)$. Finally, let's graph $F(1/2, 12, x, 30)$ (so that $ab > 1 + \frac{3}{2}\pi = 5.712388981$) as follows. $F(1/2, 12, x, 30)$




#

This looks like what we want. Therefore, from this worksheet, we learn that to make a highly oscillating trigometric function, such as $\sum_{k=1}^{\infty} a^k \cos b^k \pi x$, to be nowhere differentiable, the key is not to increase its partial sum but to increase the ratio of $\frac{b}{a}$.

# Animation

We link to Maple, click here.

**Remark** We should keep students informed that they are only looking at the graph of a partial sum, also it does not represent the graph of a real-valued function (in the sense that computer algebra system only uses certain number of points to graph a function.) Nonetheless, by incorporating the techniques of observing the ratio of $\frac{b}{a}$ and increase the partial sums, students are able to absorb a complex idea more at ease.

1 (bibitem) Körner, T.W., *Fourier Analysis*, page 38-41, Cambridge University Press, 1988.