

Countable and Uncountable Sets

What follows is a different, and I hope simpler, presentation of the material in Appendix 3 of Grimaldi. For practice problems you should do Exercises A-3, page A-34, 1-6, 8. All of these can be done using the techniques and results described here. Answers to the odd numbered exercises are in the text. Answers to the even numbered exercises are at the end of this document.

Another name for a 1-1 and onto function (or, bijection) from A to B is a *1-1 correspondence from A to B* . The inverse of such a function is a 1-1 correspondence from B to A . Thus, we often simply say that *A and B can be put into 1-1 correspondence* if there is a 1-1 and onto function from A to B . Equivalently, A and B can be put into 1-1 correspondence if there is 1-1 and onto function from B to A .

Example 1. The function $f(n) = n + 1$ is a 1-1 correspondence from \mathbf{N} to \mathbf{Z}^+ . Thus, the positive integers and the natural numbers can be put into 1-1 correspondence.

Any two finite sets with the same number of elements can be put into 1-1 correspondence. Conversely, if A and B are finite sets that can be put into 1-1 correspondence, then we know A and B have the same number of elements. In what follows, the idea of the “size” of a set is extended to infinite sets.

We say that two sets A and B have the same *cardinality* (or, size), and write $|A| = |B|$ if they can be put into 1-1 correspondence. We say that a set A has *cardinality n* , and write $|A| = n$, if A can be put into 1-1 correspondence with $\{1, 2, \dots, n\}$ for some natural number n .

The cardinality of a finite set is just the number of elements it contains, so the definition agrees with our previous use of the notation $|A|$. Notice that $\{1, 2, \dots, n\}$ is empty when $n = 0$.

If $|A| = n > 0$ then there is a 1-1 correspondence $f : \{1, 2, \dots, n\} \rightarrow A$. Thus, the elements of A can be listed as a finite sequence $f(1), f(2), \dots, f(n)$. Conversely, if there is a finite sequence x_1, x_2, \dots, x_n in which each element of A appears exactly once, then there is a 1-1 correspondence $f : \{1, 2, \dots, n\} \rightarrow A$: define $f(i) = x_i$, $1 \leq i \leq n$.

If $|A| = |\mathbf{Z}^+|$, then there is a 1-1 correspondence $g : \mathbf{Z}^+ \rightarrow A$, so the elements of A can be listed as an infinite sequence $g(1), g(2), g(3), \dots$. Conversely, if there is an infinite sequence x_1, x_2, x_3, \dots in which each element of A appears exactly once, then there is a 1-1 correspondence $f : \mathbf{Z}^+ \rightarrow A$: define $f(i) = x_i$, $i = 1, 2, \dots$.

A set A is called *countably infinite* if it can be put into 1-1 correspondence with \mathbf{Z}^+ . A set is called *countable* if it is finite or countably infinite. A set which is not countable is called *uncountable*.

The uncountable sets are the infinite sets which can not be put into 1-1 correspondence with the positive integers. The real numbers is one example of such a set. The existence of uncountable sets means that not all infinite sets have the same cardinality.

Example 1 shows that the set of natural numbers is countably infinite.

Example 2. The function $f : \mathbf{Z} \rightarrow \mathbf{Z}^+$ defined by

$$f(n) = \begin{cases} 2n, & \text{if } n > 0; \\ 2(-n) + 1 & \text{if } n \leq 0 \end{cases}$$

is a 1-1 correspondence (exercise: prove it). Thus, \mathbf{Z} is countable.

To prove that a set A is countable, it must be shown that A finite, or that A can be put into 1-1 correspondence with \mathbf{Z}^+ . We currently have three methods for performing the latter task:

- (1) Directly (by describing a function) as in Examples 1 and 2 above.
- (2) Write a sequence in which every element of A appears exactly once, as explained above.
- (3) Find a 1-1 correspondence between A and some other set which is known to be countably infinite.

Here's why method (3) works: Suppose the set X is known to be countable. Then, there is a 1-1 and onto function $g : \mathbf{Z}^+ \rightarrow X$. If A can be put into 1-1 correspondence with X , then there is a 1-1 and onto function $f : X \rightarrow A$. The composite function $f \circ g$ is then a 1-1 and onto function from \mathbf{Z}^+ to A because f and g are both 1-1 and onto.

In the next few results, we will develop a couple of other methods for showing that a set is countable.

Fact 1. *Any subset of \mathbf{N} is countable.*

Proof. Let $A \subseteq \mathbf{N}$. If A is finite, it is countable by definition. Otherwise, there is an infinite sequence x_1, x_2, \dots in which every element of A appears exactly once: let x_1 be the smallest element in A , and for $i > 1$, let x_i be the smallest element in the set $A - \{x_1, x_2, \dots, x_{i-1}\}$. ■

Example 3. By Fact 1 the set of prime numbers $\{2, 3, 5, 7, 11, 13, 17, \dots\}$ is countable, since it is a subset of \mathbf{N} . The same is true for the set of all squares of integers, $\{1, 4, 9, \dots, n^2, \dots\}$, the set of all even non-negative integers, $\{2n : n \in \mathbf{N}\}$, and much more.

Fact 2. *Any subset of the negative integers is countable.*

Proof. Exercise. ■

Fact 3. *If A and B are countable sets, then so is $A \cup B$.*

Proof. If A and B are both finite, then so is $A \cup B$, and any finite set is countable. Otherwise, let $B' = B - A$, so that A and B' have no elements in common.

Suppose that A is finite and B' is countably infinite. Then, there is a finite sequence a_1, a_2, \dots, a_n that contains every element of A exactly once, and an infinite sequence b_1, b_2, \dots that contains every element of B' exactly once. An infinite sequence that contains every element of $A \cup B$ (I really mean B - the elements of $A \cap B$ are listed among the elements of A) exactly once is $a_1, a_2, \dots, a_n, b_1, b_2, \dots$

A similar argument works if A is countably infinite and B' is finite.

Finally, suppose both A and B' are countably infinite. Then, there is an infinite sequence a_1, a_2, \dots that contains every element of A exactly once, and an infinite sequence b_1, b_2, \dots that contains every element of B' exactly once. An infinite sequence that contains every element of $A \cup B$ exactly once is $a_1, b_1, a_2, b_2, \dots$ ■

Corollary 4. *Any subset of the integers is countable.*

Proof. Let $X \subseteq \mathbf{Z}$. Let $A = X \cap \mathbf{N}$ and $B = X - \mathbf{N}$. Then A is countable by Fact 1, and B is countable by Fact 2. Therefore $X = A \cup B$ is countable by Fact 3. ■

Example 4. By Corollary 4 the set of all even integers is countable, as is the set of all multiples of three, the set of all cubes of integers, etc.

It follows from Corollary 4 that once a set can be put into 1-1 correspondence with any subset of the integers, it is countable. This leads to another method for proving that a set is countable:

Fact 5. *Let A be an infinite set. If there is an infinite sequence in which each element of A appears at least once, then A is countable.*

Proof. Let x_1, x_2, \dots be an infinite sequence in which each element of A appears at least once. Define $f : A \rightarrow \mathbf{Z}^+$ by $f(a)$ is the smallest index i such that $a = x_i$. Since each element of A appears at least once in the sequence, the positive integer i exists. Let $B = f(A)$ be the range of f . Then $B \subseteq \mathbf{Z}^+$, so B is countable. Now, f maps A onto B by the definition of the range and furthermore, f is 1-1: if $f(x) = f(y) = i$, then $x = y = x_i$. Thus, f is a 1-1 correspondence between A and a countable set. Therefore, A is countable. ■

The proof of Fact 5 uses the fact that the existence of a 1-1 function $f : A \rightarrow \mathbf{Z}^+$ implies the existence of a 1-1 correspondence from A to a subset of \mathbf{Z}^+ . This implies that A is countable since there is a 1-1 correspondence between A and a subset of the integers, and any subset of \mathbf{Z} is countable. Thus, the proof of Fact 5 shows that if there is a 1-1 function $f : A \rightarrow \mathbf{Z}^+$, then the set A is countable (so the requirement that f be onto can be dropped). Since any countably infinite set X can be put into 1-1 correspondence with \mathbf{Z}^+ , the set \mathbf{Z}^+ can be replaced by X in the above argument. Therefore, a set A is countably infinite if there is a 1-1 function from A to *any* countable set. This makes it really easy to prove the following generalisation of Corollary 4.

Fact 6. *Any subset of a countable set is countable.*

Proof. Let A be a countable set and $X \subseteq A$. If X is finite, it is countable. Suppose X is infinite. Then the function $f : X \rightarrow A$ defined by $f(x) = x$ is 1-1. Therefore X is countable. ■

Fact 5 can be used to prove the following generalisation of Fact 3.

Fact 7. *If A_1, A_2, \dots is a collection of countably many countably infinite sets, then*

$$\bigcup_{i=1}^{\infty} A_i$$

is also a countable set.

Proof. For each positive integer i there is an infinite sequence $a_{i,1}, a_{i,2}, \dots$ that contains every element of A_i exactly once. Then $a_{1,1}, a_{1,2}, a_{2,1}, a_{1,3}, a_{2,2}, a_{3,1}, a_{1,4}, a_{2,3}, \dots$ is an infinite sequence that contains every element of $\bigcup_{i=1}^{\infty} A_i$ at least once. (The elements $a_{i,j}$ are

listed in non-decreasing order of $i + j$, and if $i + j = r + s$ then $a_{i,j}$ is listed before $a_{r,s}$ if and only if $i < r$.) By Fact 4, $\bigcup_{i=1}^{\infty} A_i$ is countable. ■

The technique used in the above proof is called *diagonal sweeping*, since it corresponds to writing the sequences in a 2-dimensional array and then constructing the sequence by sweeping down consecutive back diagonals starting from the top left corner. Exactly the same argument can be used to prove the following.

Example 5. *The set \mathbf{Q} of rational numbers is countable.*

For each positive integer i , let A_i be the set of rational numbers with denominator equal to i . Then, every element of A_i occurs exactly once in the sequence $\{a_{i,j}\}_{j=1}^{\infty} = 0/i, 1/i, -1/i, 2/i, -2/i, \dots$. Thus, A_i is countable. Since $\mathbf{Q} = \bigcup_{i=1}^{\infty} A_i$, the set \mathbf{Q} of rational numbers is countable by Fact 7. ■

Uncountable sets do exist however. For example:

Fact 8. *The set S of all infinite binary sequences is uncountable.*

Proof. The proof is by contradiction. Suppose that S is countable. Since S is clearly not a finite set, this means that there is a 1-1 correspondence $f : \mathbf{Z}^+ \rightarrow S$. Hence, every infinite binary sequence $\sigma \in S$ is $f(n)$ for some positive integer n . We will obtain a contradiction to the fact that f is onto by exhibiting an element τ of S that, by its very definition, can not be $f(n)$ for any positive integer n . For each positive integer i , let $f(i)$ be the sequence $f_{i,1}, f_{i,2}, \dots$. Then, the infinite binary sequence $\tau = t_1, t_2, t_3, \dots$ is defined by $t_i = 1 - f_{i,i}$, for $i = 1, 2, \dots$

We claim that τ can not be $f(n)$ for any positive integer n . This is true because, for every positive integer n , the n -th element of the sequence τ is different from $f_{n,n}$, the n -th element of $f(n)$. This establishes the contradiction mentioned above, and therefore there can not be a 1-1 correspondence from \mathbf{Z}^+ to S . Hence, S is uncountable. ■

The technique used in the proof of Fact 8 is known as “Cantor Diagonalization” because the sequence τ arises from changing the “diagonal elements” $f_{i,i}$.

The technique of Cantor Diagonalization can also be used to prove that the set of real numbers is uncountable. Before we do this, we need to review a concept. Every real number has an infinite decimal expansion. Most real numbers have only one of these, but some have two. Any number with a finite decimal expansion has one infinite decimal expansion that ends in an infinite sequence of zeros and a second infinite decimal expansion that ends in an infinite sequence of nines. For example, $3.2700000\dots = 3.269999\dots$. These are the only numbers with more than one infinite decimal expansion. We will need to make use of this in our proof.

Fact 9. *The set \mathbf{R} of real numbers is uncountable.*

Proof. The set \mathbf{R} is clearly infinite. Suppose it is countable. Then, there exists a 1-1 correspondence $f : \mathbf{Z}^+ \rightarrow \mathbf{R}$. We obtain a contradiction by finding $x \in \mathbf{R}$ such that x is not $f(n)$ for any positive integer n .

For each positive integer i , suppose that the part of the decimal expansion of $f(i)$ following the decimal point is $.d_{i,1}d_{i,2}d_{i,3}\dots$, where each $d_{i,j}$ is one of $0, 1, \dots, 9$. The real

number $x = 0.x_1x_2x_3\dots$ is defined by

$$x_i = \begin{cases} 5 & \text{if } d_{i,i} \neq 5 \\ 6 & \text{if } d_{i,i} = 5 \end{cases}$$

for each positive integer i .

We claim that x can not be $f(n)$ for any positive integer n . First of all, x differs from $f(n)$ in the the n -th digit after the decimal point (i.e., $x_i \neq d_{i,i}$). And, secondly, x is a number with only one infinite decimal expansion so it can not simply be a second representation of a number that is $f(n)$ for some n . This proves the claim.

Therefore, f is not onto, a contradiction. Hence, \mathbf{R} is uncountable. ■

Since the real numbers are uncountable and the rational numbers are countable, it follows that irrational numbers exist. It is not difficulty to show (but you don't have to do it) that $n^{1/k}$ is irrational unless n is the k -th power of some integer. In particular, $\sqrt{2}$ is irrational. A slightly more advanced argument can be used to show that e , the base of the natural logarithms, is irrational, and a much more complicated argument is required to show that π is irrational. So far we have described only countably many irrational numbers. But the set of irrational numbers must be uncountable because otherwise \mathbf{R} would be the union of two countable sets, and would therefore be countable.

It is unknown whether there is a set whose cardinality lies "between" $|\mathbf{Z}^+|$ and $|\mathbf{R}|$. This is known as *the continuum hypothesis* partly because the cardinality of the real numbers is sometimes called the continuum. The continuum hypothesis is known to be beyond proof in our system of logic.

Since $\mathbf{Z}^+ \subseteq \mathbf{R}$, we would be justified in writing $|\mathbf{Z}^+| < |\mathbf{R}|$. However, you should not believe that the continuum is the largest possible cardinality of an infinite set. There is an infinitely heirarchy of cardinalities of infinite sets because of the following fact.

Fact 10. For any set A , $|A| < |\mathcal{P}(A)|$.

Proof. First of all, $|A| \leq |\mathcal{P}(A)|$, since A can clearly be put into 1-1 correspondence with the subset of $\mathcal{P}(A)$ consisting of the singleton subsets, $\{\{x\} : x \in A\}$.

Suppose $|A| = |\mathcal{P}(A)|$. Then there is a 1-1 correspondence $f : A \rightarrow \mathcal{P}(A)$. We obtain a contradiction to the fact that f is onto by exhibiting a subset X of A such that $X \neq f(a)$ for any $a \in A$.

For every $a \in A$, either $a \in f(a)$, or $a \notin f(a)$. The definition of X is slippery: let $X = \{a \in A : a \notin f(a)\}$.

Consider $a \in A$. If $a \in f(a)$ then $a \notin X$ (by definition of X), so $f(a) \neq X$. On the other hand, if $a \notin f(a)$ then $a \in X$ (by definition of X), so $f(a) \neq X$. Therefore, $X \neq f(a)$ for any $a \in A$, a contradiction. Hence, $|A| < |\mathcal{P}(A)|$. ■

By Fact 10, $|\mathbf{R}| < |\mathcal{P}(\mathbf{R})| < |\mathcal{P}(\mathcal{P}(\mathbf{R}))| < \dots$

Solutions to even numbered problems in Grimaldi

2. (a) $f(n) = n^2$. (b) $f(n) = 4n + 2$.

4. The set I is uncountable. If I were countable then \mathbf{R} would be the union of two countable sets and therefore countable, a contradiction.

6. There is an infinite sequence in which each element of $\mathbf{Z}^+ \times \mathbf{Z}^+ \times \mathbf{Z}^+$ appears exactly once: $(1, 1, 1), (1, 1, 2), (1, 2, 1), (2, 1, 1), (1, 1, 3), \dots$. The elements are ordered by non-decreasing sum of the components. The elements whose components have the same sum appear in dictionary order.

8. (a) $f(x) = 3x$. (b) $f(x) = 5x + 2$. (c) $f(x) = (b - a)x + a$

More Problems

9. Show that if A_1, A_2, \dots is a countable collection of finite sets, then $X = \cup_{i=1}^{\infty} A_i$ is a countable set.

(Hint: when X is infinite, use Fact 5.)

10. Show that the set of all binary sequences of finite length is a countable set.

11. Let Σ be a finite set. Show that the set of all sequences of elements of Σ of finite length is countable.

12. Argue that the set of all computer programs is a finite set.

13. (Harder) Show that if A_1, A_2, \dots is a countable collection of countable sets, then $X = \cup_{i=1}^{\infty} A_i$ is a countable set.

(Hint: First use number 10 above to combine the finite sets into one countable set, then use Fact 7 to do the same for the countably infinite sets. Finally, if necessary, use Fact 3.)