# Section 7.1

The Pythagorean Theorem

The origins of right triangle geometry can be traced back to 3000 BC in Ancient Egypt. The Egyptians used special right triangles to survey land by measuring out 3-4-5 and 5-12-13 right triangles to make right angles. The Egyptians mostly understood right triangles in terms of ratios or what would now be referred to as Pythagorean Triples. The Egyptians also had not developed a formula for the relationship between the sides of a right triangle. At this time in history, it is important to know that the Egyptians also had not developed the concept of a variable. The Egyptians most studied specific examples of right triangles. For example, the Egyptians use ropes to measure out distances to form right triangles that were in whole number ratios. In the next illustration, it is demonstrated how a 3-4-5 right triangle can be form using ropes to create a right angle.
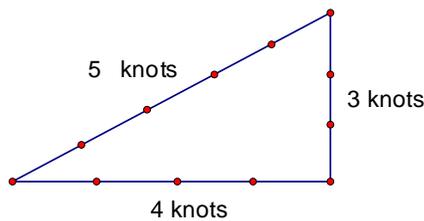

Using ropes that had knots that where equality spaced, the Egyptians could measure out right angles by making a 3-4-5 right angle or other right triangles with the rope.

It wasn't until around 500 BC, when a Greek mathematician name Pythagoras discovered that there was a formula that described the relationship between the sides of a right triangle. This formula was known as the Pythagorean Theorem.
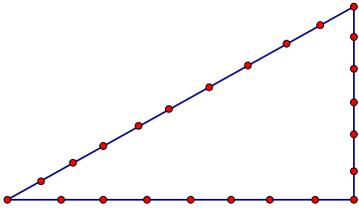
Pythagorean Theorem

In a right triangle, the sum of the squares of the two legs is equal to the square of the hypotenuse.
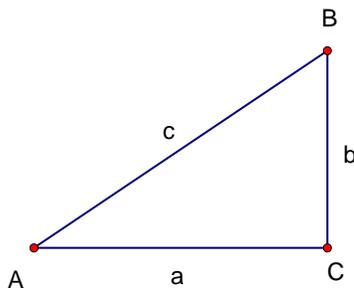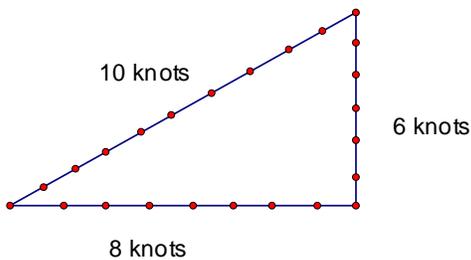

$$c^2 = a^2 + b^2$$

Example 1

Determine if the triangle measured out by ropes has out a right angle.

If you count the number of knots on each side of the triangle you get a ratio of 6-8-10.

10 knots

6 knots

8 knots

Substituting these values into the Pythagorean Theorem using 10 as the hypotenuse and the other two sides as the legs, you can determine if the triangle is a right triangle.

$$c^2 = a^2 + b^2$$
$$10^2 = 6^2 + 8^2$$
$$10 = 36 + 64$$
$$100 = 100$$

Since the formula checks, the triangle is a right triangle which gives us a right angle.

Here are some examples of how the Pythagorean Theorem can be use to find the missing side of a right triangle.

Example 2

Suppose the two legs of a right triangle are 5 units and 12 units, find the length of the hypotenuse.

To find the solution, substitute the value of the legs into the Pythagorean Theorem and solve for the hypotenuse.

Let $a = 5$ and $b = 12$, and solve for c

$$c^2 = 5^2 + 12^2$$

$$c^2 = 25 + 144$$

$$c^2 = 169$$

$$\sqrt{c^2} = \sqrt{169}$$

$$c = 13$$

Example 3

Suppose that the hypotenuse of a right triangle is 26 units and one leg is 10 units, find the measure of the other leg

To find the solution, substitute the value of the leg and hypotenuse into the Pythagorean Theorem and solve for the missing leg.

Given $a = 10, c = 26$, find b

$$26^2 = 10^2 + b^2$$

$$676 = 100 + b^2$$

$$676 - 100 = 100 - 100 + b^2$$

$$576 = b^2$$

$$\sqrt{b^2} = \sqrt{575}$$

$$b = 24$$

Applications of the Pythagorean Theorem

The Pythagorean Theorem has several real life applications. This is due to the fact that so many problems can be model or represented by a right triangle. If this is the case, then values can be assigned to the sides of the triangle and the unknown value can be found by solving for the missing side of the triangle. Here are some examples of applications of right triangles and the Pythagorean Theorem

Example 4

An empty lot is 120 ft by 50 ft. How many feet would you save walking diagonally across the lot instead of walking length and width?
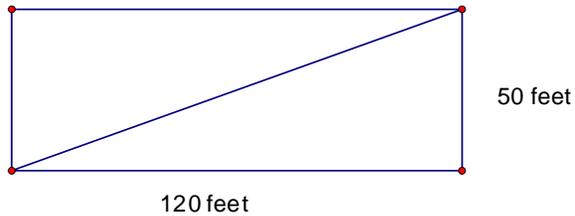
50 feet

120 feet

$$c^2 = 120^2 + 50^2$$
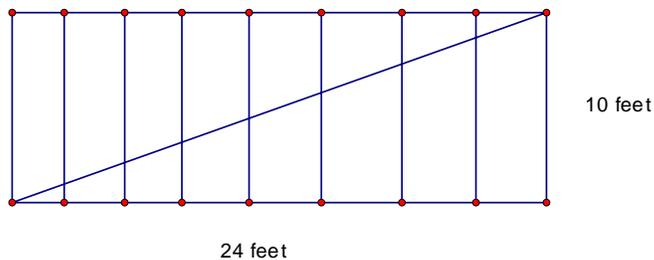$$c^2 = 14400 + 2500$$
$$c^2 = 16900$$
$$\sqrt{c^2} = \sqrt{16900}$$
$$c = 130\, ft$$

Compared to walking 120 ft + 50 ft = 170 ft

You would save walking 170 ft – 130 ft = 40 feet

Example 5

A diagonal brace is to be placed in the wall of a room. The height of the wall is 10 feet and the wall is 24 feet long. (See diagram below) What is the length of the brace?
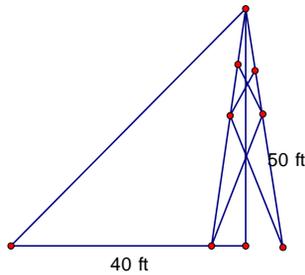

10 feet

24 feet

$$c^2 = 10^2 + 24^2$$
$$c^2 = 100 + 576$$
$$c^2 = 676$$
$$\sqrt{c^2} = \sqrt{676}$$
$$c = 26\, feet$$

Example 6

A television antenna is to be erected and held by guy wires. If the guy wires are 40 ft from the base of the antenna and the antenna is 50 ft high, what is the length of each guy wire?

50 ft

40 ft

$$c^2 = 40^2 + 50^2$$

$$c^2 = 1600 + 2500$$

$$c^2 = 4100$$

$$\sqrt{c^2} = \sqrt{4100} \Rightarrow c \approx 64\,feet$$

Math History Excursion: Commensurability and the Pythagorean Theorem

The Pythagoreans as well as the Greeks believed that all distances and measurements were commensurable. If two line segments a and b are **commensurable**, then there exist a third segment that can be laid end-to-end a whole number of times to produce segments that are equal in length to both line segment a and line segment b.

Definition: If two segments are commensurable, then there exists a third line segment c such that $a = mc$ and $b = nc$ where n and m are integers.

For example, let's use segments that measure 3 inches and 5 inches in length. These segments are commensurable, because there exist a third segment that measures .5 inches in length that can laid end-to-end to 6 times to produce the 3 inch segment and 10 ten times to produce the 5 inch segment. (See illustration below)

a = 5 inches

b = 3 inches

c=.5 inches

segment c laid out 6 times is 3 inches

segment c laid out 10 times is 5 inches

It is also true that two lengths are commensurable only if their quotient is a rational number. A **rational number** is a number that can be expressed as the quotient of two integers. A number that can not be expressed as a quotient of integers or a fraction is called an **irrational number**. Also recall that the set of **real numbers** can be broken down into two separate or disjoints sets which are the rational numbers and the irrational numbers.

The development of the Pythagorean Theorem invalidated the Greek and Pythagorean notion that all measurements are commensurable. For example, suppose that we have square with side measuring 1 unit. If we use the Pythagorean Theorem to find the diagonal or hypotenuse, we get that the hypotenuse is $\sqrt{2}$. (See illustration below)
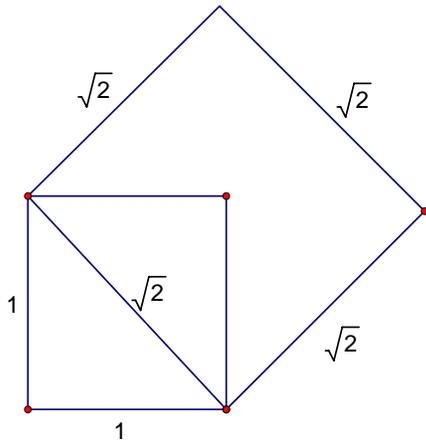
$$c^2 = 1^2 + 1^2$$
$$c^2 = 1 + 1$$
$$c^2 = 2$$
$$\sqrt{c^2} = \sqrt{2}$$
$$c = \sqrt{2}$$

This general observation produces two segments with lengths 1 and $\sqrt{2}$ that are not commensurable or **incommensurable**. Notice, if 1 and $\sqrt{2}$ are commensurable then their quotient would be rational. This would imply that $\dfrac{\sqrt{2}}{1} = \sqrt{2}$ is rational which is obviously false. Therefore, $\sqrt{2}$ or any other irrational number is going to produce a set of numbers that are incommensurable. This fact alone was a contraction to Greek and Pythagorean concept that all measurements where commensurable. However, even when the Pythagoreans later discovered that the hypotenuse of a right triangle could equal $\sqrt{2}$, they initially believed that $\sqrt{2}$ was somehow rational. In fact, the Pythagoreans believed that mathematics and religion were one. They also believe that all natural phenomena could be expressed by whole numbers or ratios of whole numbers. For this reason, they

believed that $\sqrt{2}$ could expressed as a ratio of whole numbers. It is believed that it was Hippasus of Metapontum, a Pythagorean, who later discovered that number could be irrational while identifying the sides of the pentagram. Later, Theodorus of Cyrene proved that certain numbers where irrational, but it wasn't until Exdoxus developed a theory of irrational ratios that a strong mathematical foundation for irrational numbers was created. To finish up this section, here are some examples of rational numbers and irrational numbers.

Examples of Rational Numbers

$$\frac{2}{5}, 3, -2, 0, .25, .3333......, \frac{1}{3}$$

Note: 3 is rational because can be written as $\frac{3}{1}$ and .3333….. is equal to $\frac{2}{3}$

Also not that any decimal number that repeats or terminates is rational.

Examples of Irrational Numbers

$$\sqrt{2}, \sqrt{11}, \sqrt{5}, \pi, e$$

Note: The value of pi π and the Euler number e are irrational.

Example 6

Describe each number in the list as irrational or rational.

$$\sqrt{7}, -1, \pi, \frac{2}{3}$$

Solution:

Irrational numbers $\sqrt{7}, \pi$

Rational numbers $-1, \frac{2}{3}$

Exercises

1) Find the missing side of right triangle ABC where a and b are the legs and c is the hypotenuse.

    a) a = 10, b = 24, find c
    b) a = 12, c = 20, find b
    c) a = 8, b = 8, find c
   d) a = 5, b = 12, find c

2) Suppose that the hypotenuse of a right triangle is 10 units and one leg is 8 units, find the measure of the other leg

3) Suppose the two legs of a right triangle are 5 units and 7 units respectively, find the length of the hypotenuse.

4) A rectangular shaped lot is 80 ft by 60 ft. How many feet would you save walking diagonally across the lot instead of walking length and width?

5) Find the length diagonal of a rectangle that is 30 ft by 40 ft.

6) A diagonal brace is to be placed in the wall of a room. The height of the wall is 8 feet and the wall is 20 feet long. (See diagram below) What is the length of the brace?

8 feet

20 feet
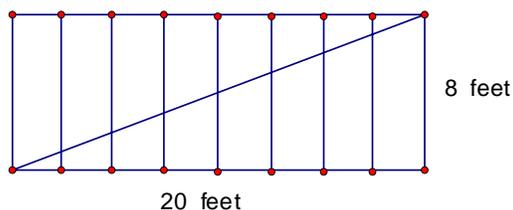
7) A television antenna is to be erected and held by guy wires. If the guy wires are 40 ft from the base of the antenna and the antenna is 70 ft high, what is the length of the guy wire?

h = 70 feet

40 feet   40 feet