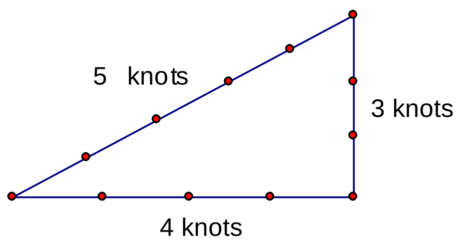
Math 135
Section 3.3

Right Triangles

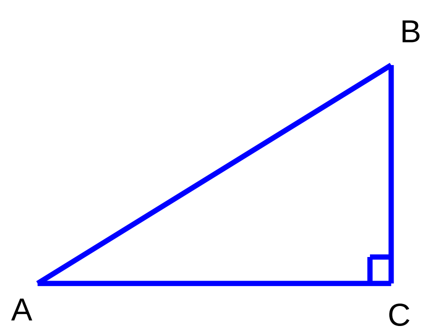
The origins of right triangle geometry can be traced back to 3000 BC in Ancient Egypt. The Egyptians used special right triangles to survey land by measuring out 3-4-5 right triangles to make right angles. The Egyptians mostly understood right triangles in terms of ratios or what would now be referred to as Pythagorean Triples. The Egyptians also had not developed a formula for the relationship between the sides of a right triangle. At this time in history, it is important to know that the Egyptians also had not developed the concept of a variable or equation. The Egyptians most studied specific examples of right triangles. For example, the Egyptians use ropes to measure out distances to form right triangles that were in whole number ratios. In the next illustration, it is demonstrated how a 3-4-5 right triangle can be form using ropes to create a right angle.



It wasn't until around 500 BC, when a Greek mathematician name Pythagoras discovered that there was a formula that described the relationship between the sides of a right triangle. This formula was known as the Pythagorean Theorem.

Pythagorean Theorem

In a right triangle, the sum of the squares of the two legs is equal to the square of the hypotenuse.

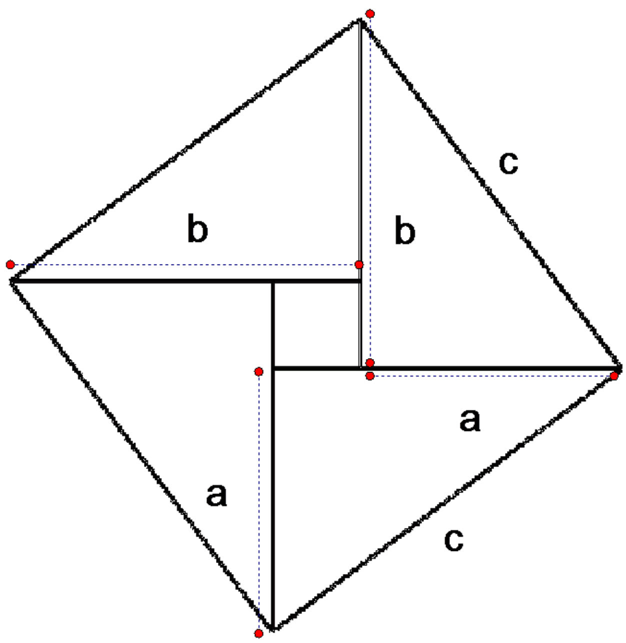


$c^2 = a^2 + b^2$

The Pythagorean Theorem

Use the following picture to derive the Pythagorean Theorem. $\left(c^2 = a^2 + b^2\right)$

Behold!



Area of the square = Area of small square + Area of 4 triangles

$$c^2 = (b-a)^2 + 4\left(\frac{1}{2}ab\right)$$

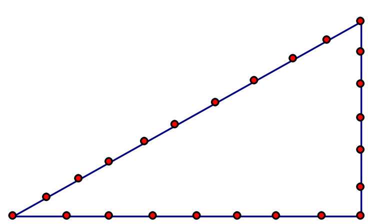
$$c^2 = (b-a)(b-a) + 2(ab)$$

$$c^2 = b^2 - 2ab + a^2 + 2ab$$

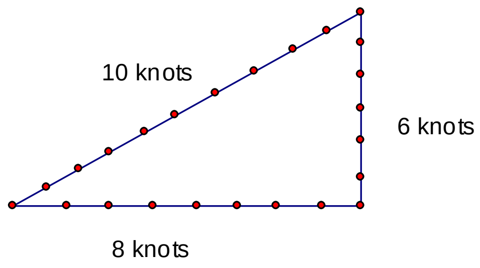
$$c^2 = a^2 + b^2$$

Example 1

Determine if the triangle measured out by ropes has out a right angle.



If you count the number of knots on each side of the triangle you get a ratio of 6-8-10.



Substituting these values into the Pythagorean Theorem using 10 as the hypotenuse and the other two sides as the legs, you can determine if the triangle is a right triangle.

$$c^2 = a^2 + b^2$$
$$10^2 = 6^2 + 8^2$$
$$10 = 36 + 64$$
$$100 = 100$$

Since the formula checks, the triangle is a right triangle which gives us a right angle.

Here are some examples of how the Pythagorean Theorem can be use to find the missing side of a right triangle.

Example 2

Suppose the two legs of a right triangle are 5 units and 12 units, find the length of the hypotenuse.

To find the solution, substitute the value of the legs into the Pythagorean Theorem and solve for the hypotenuse.

$$Let\ a = 5\ and\ b = 12,\ and\ solve\ for\ c$$
$$c^2 = 5^2 + 12^2$$
$$c^2 = 25 + 144$$
$$c^2 = 169$$
$$\sqrt{c^2} = \sqrt{169}$$
$$c = 13$$

Example 3

Suppose that the hypotenuse of a right triangle is 26 units and one leg is 10 units, find the measure of the other leg

To find the solution, substitute the value of the leg and hypotenuse into the Pythagorean Theorem and solve for the missing leg.

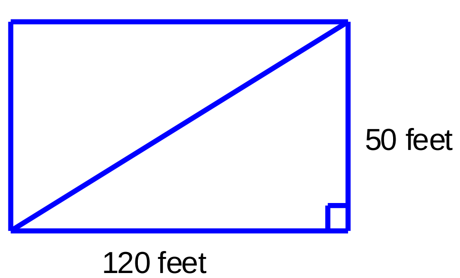
$$\begin{aligned}
&Given\ a = 10, c = 26,\ \ find\ \ b \\
&26^2 = 10^2 + b^2 \\
&676 = 100 + b^2 \\
&676 - 100 = 100 - 100 + b^2 \\
&576 = b^2 \\
&\sqrt{b^2} = \sqrt{575} \\
&b = 24
\end{aligned}$$

Applications of the Pythagorean Theorem

The Pythagorean Theorem has several real life applications. This is due to the fact that so many problems can be model or represented by a right triangle. If this is the case, then values can be assigned to the sides of the triangle and the unknown value can be found by solving for the missing side of the triangle. Here are some examples of applications of right triangles and the Pythagorean Theorem

Example 4

An empty lot is 120 ft by 50 ft. How many feet would you save walking diagonally across the lot instead of walking length and width?



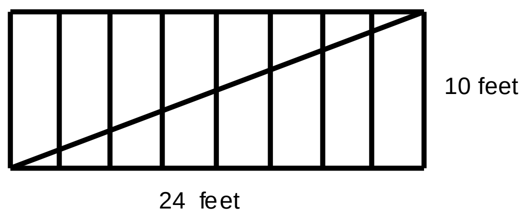
$$c^2 = 120^2 + 50^2$$
$$c^2 = 14400 + 2500$$
$$c^2 = 16900$$
$$\sqrt{c^2} = \sqrt{16900}$$
$$c = 130\ \mathit{ft}$$

Compared to walking 120 ft + 50 ft = 170 ft

You would save walking 170 ft – 130 ft = 40 feet

Example 5

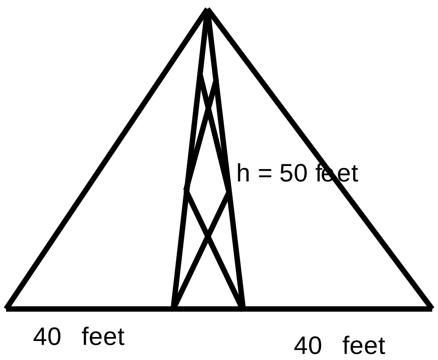
A diagonal brace is to be placed in the wall of a room. The height of the wall is 10 feet and the wall is 24 feet long. (See diagram below) What is the length of the brace?



$$c^2 = 10^2 + 24^2$$
$$c^2 = 100 + 576$$
$$c^2 = 676$$
$$\sqrt{c^2} = \sqrt{676}$$
$$c = 26\ \mathit{feet}$$

Example 6

A television antenna is to be erected and held by guy wires. If the guy wires are 40 ft from the base of the antenna and the antenna is 50 ft high, what is the length of each guy wire?



$c^2 = 40^2 + 50^2$

$c^2 = 1600 + 2500$

$c^2 = 4100$

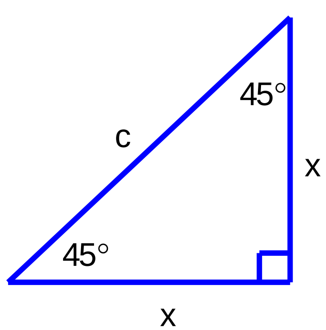
$\sqrt{c^2} = \sqrt{4100} \Rightarrow c \approx 64\, feet$

Special Triangles

The $45^0 - 45^0 - 90^0$ triangle

In a $45^0 - 45^0 - 90^0$ triangle, the hypotenuse is equal to $\sqrt{2}$ times the length of the length of the leg.

Proof:



$c^2 = x^2 + x^2$

$c^2 = 2x^2$

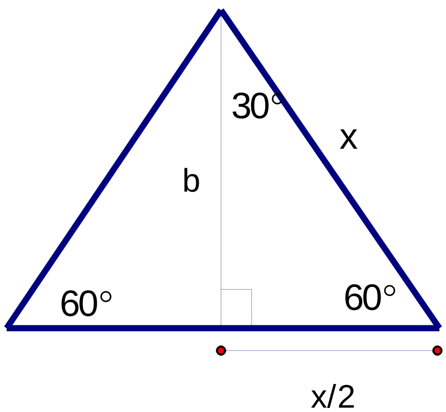
$\sqrt{c^2} = \sqrt{2x^2}$

$c = x\sqrt{2}$

The $30^0 - 60^0 - 90^0$ Triangle

If a $30^0 - 60^0 - 90^0$ triangle, the length of the hypotenuse is twice the shorter legs and the length of the longer leg is $\sqrt{3}$ times the length the shorter leg.

Proof:
Start with an equilateral triangle with a side of length x and construct a diagonal to the base of the triangle. This will make two $30^0 - 60^0 - 90^0$ triangles with a base measuring $\frac{x}{2}$



Now, use the Pythagorean Theorem to find the missing side.

$$c^2 = a^2 + b^2$$
$$x^2 = \left(\frac{x}{2}\right)^2 + b^2$$
$$x^2 = \frac{x^4}{4} + b^2$$
$$b^2 = x^2 - \frac{x^2}{4}$$
$$b^2 = \frac{3x^2}{4}$$
$$\sqrt{b^2} = \sqrt{\frac{3x^2}{4}} \Rightarrow b = x\frac{\sqrt{3}}{2}$$

Since the longer side is $b = x\frac{\sqrt{3}}{2}$, the longer side would be $\sqrt{3}$ times the shorter side