Convergence of the Secant Method

Here are my calculations for the secant method. They differ from those on page 144 in two ways. They avoid use of the top equation on page 144. They also address exercise 4.7 on page 150 and show that the order of convergence of the secant method is

$$r = \frac{1+\sqrt{5}}{2} \sim 1.618.$$

The claim is that e_{n+1} is proportional to the product $e_n\, e_{n-1}$. Okay then, the secant method is given by

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}.$$

Here's an exercise for you. Show that

$$e_{n+1} = \frac{f(x_n)\, e_{n-1} - f(x_{n-1})\, e_n}{f(x_n) - f(x_{n-1})}.$$

This may be rewritten as

$$e_{n+1} = e_n\, e_{n-1} \left(\frac{\dfrac{f(x_n)}{e_n} - \dfrac{f(x_{n-1})}{e_{n-1}}}{f(x_n) - f(x_{n-1})} \right).$$

Since $f(\alpha) = 0$ we can rewrite this as

$$e_{n+1} = e_n\, e_{n-1} \left(\frac{\dfrac{f(x_{n-1}) - f(\alpha)}{x_{n-1} - \alpha} - \dfrac{f(x_n) - f(\alpha)}{x_n - \alpha}}{f(x_n) - f(x_{n-1})} \right) \tag{1}$$

Concentrate on the numerator and define

$$F(x) = \frac{f(x) - f(\alpha)}{x - \alpha}.$$

The Mean Value Theorem says there is a ζ_n between x_{n-1} and x_n for which

$$F(x_{n-1}) - F(x_n) = F'(\zeta_n)(x_{n-1} - x_n).$$

But, as you learned in Math 152,

$$F'(x) = \frac{f'(x)(x-\alpha) + f(\alpha) - f(x)}{(x-\alpha)^2} \tag{2}$$

Taylor's Theorem for f expanded about x says there is a ν between α and x for which

$$f(\alpha) = f(x) + f'(x)(\alpha - x) + f''(\nu_n)\frac{(\alpha - x)^2}{2}$$

which gives

$$f(\alpha) - \{f(x) + f'(x)(\alpha - x)\} = f''(\nu_n)\,\frac{(\alpha - x)^2}{2} \qquad (3)$$

Here's another exercise for you. Combine equations (1), (2), and (3) to show that

$$F(x_{n-1}) - F(x_n) = \frac{x_{n-1} - x_n}{2} f''(\nu_n).$$

Notice that the left side is nothing more than the negative of the numerator in equation (1); so (1) becomes

$$e_n\, e_{n-1} \left(\frac{\dfrac{f(x_{n-1}) - f(\alpha)}{x_{n-1} - \alpha} - \dfrac{f(x_n) - f(\alpha)}{x_n - \alpha}}{f(x_n) - f(x_{n-1})} \right) = -\frac{f''(\nu_n)}{2}\,\frac{x_{n-1} - x_n}{f(x_{n-1}) - f(x_n)} = -e_{n-1}\, e_n \frac{f''(\nu_n)}{2f'(\zeta_n)}$$

which is approximately

$$-e_{n-1}\, e_n \frac{f''(\alpha)}{2f'(\alpha)}.$$

Note: As with the proof for Newton's method, the above relies on the fact that α is a simple root.

Order of Convergence for the Secant Method

Now let's look at the order of convergence.

A method is said to have order r if

$$\lim_{n \to \infty} \frac{|x_{n+1} - \alpha|}{|x_n - \alpha|^r} = C$$

for some constant C. *The question is what is* r *for the secant method?* Let

$$S_n = \frac{|e_{n+1}|}{|e_n^r|}.$$

Then

$$|e_{n+1}| = S_n |e_n^r| = S_n (S_{n-1} |e_{n-1}{}^r|)^r = S_n\, S_{n-1}^r\, |e_{n-1}|^{r^2}.$$

This shows that

$$\frac{|e_{n+1}|}{|e_n||e_{n-1}|} = \frac{S_n\, S_{n-1}^r\, |e_{n-1}|^{r^2}}{S_{n-1}\, |e_{n-1}|^r\, |e_{n-1}|} = S_n\, S_{n-1}^{r-1}\, |e_{n-1}|^{r^2 - r - 1}.$$

The first two factors in the last expression approach constants; so it must be the case that

$$r^2 - r - 1 = 0$$

which gives

$$r = \frac{1 + \sqrt{5}}{2} \sim 1.618$$

(since the other root is negative).