Quadratics

1. The Problem

For a particular species of beetle the metabolic rate increases as the temperature increases (within a particular range). The table to the right provides the oxygen consumption at certain air temperatures. The data from the table can be modeled by the quadratic function

$$C = 0.45t^2 - 1.65t + 50.75, \qquad 10 \le t \le 25$$

where t is the temperature (in degrees Celsius) and C is the oxygen consumption (in microliters per gram per hour).

Temperature	*Oxygen Consumption*
10	80
15	127
20	198
25	290

Find the air temperature when the beetle's oxygen consumption is 175 microliters per gram per hour.

In order to answer the question above, one must understand what a quadratic equation is.

2. Quadratic Equations

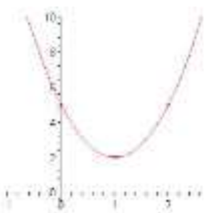
An equation in the form

$$ax^2 + bx + c = 0$$

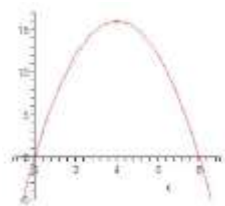
where a, b, and c are real numbers and $a \neq 0$ is called a ***quadratic equation***.

Graphically, a quadratic equation takes the form of a ***parabola***.

If $a > 0$ the parabola opens up.



If $a < 0$ the parabola opens down.



3. Solving Quadratic Equations

There are a variety of ways to solve a quadratic equation, including

(1) graphing

(2) factoring

(3) completing the square, and

(4) using the quadratic formula

We will primarily make use of the quadratic formula, but again be aware there are multiple methods to solve a quadratic equation.

Given an equation in the form $ax^2 + bx + c = 0$, solutions are calculated as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The term underneath the square root symbol is called the ***discriminant*** and whose value determines the type of solutions that exist.

- If $b^2 - 4ac > 0$ then 2 real solutions exist
  - The parabola has two x-intercepts

- If $b^2 - 4ac < 0$ then 2 complex solutions exist (the square root of a negative results in an imaginary number)
  - The parabola has no x-intercepts

- If $b^2 - 4ac = 0$ then 1 real, repeated solution exist
$$x = \frac{-b \pm \sqrt{0}}{2a} = \frac{-b \pm 0}{2a} = \frac{-b}{2a}$$
  - The parabola has one x-intercept, the vertex is on the x-axis.

Note, the quadratic equation is only used if the equation is in the form

$$a\boldsymbol{x}^{\boldsymbol{2}} + b\boldsymbol{x} + c = 0$$

(the variable, x, is arbitrary).

Ex.) Solve each equation

(1) $x^2 + x - 6 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(1) \pm \sqrt{(1)^2 - 4(1)(-6)}}{2(1)} = \frac{-1 \pm \sqrt{1 + 24}}{2} = \frac{-1 \pm \sqrt{25}}{2}$$

$$= \frac{-1 \pm 5}{2} = \begin{cases} \dfrac{-1+5}{2} = \dfrac{4}{2} = 2 \\ \text{or} \\ \dfrac{-1-5}{2} = \dfrac{-6}{2} = -3 \end{cases}$$

$x = 2$ or $x = -3$

(2) $x^2 - 4x + 2 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)} = \frac{4 \pm \sqrt{16 - 8}}{2} = \frac{4 \pm \sqrt{8}}{2}$$

$$= \frac{4 \pm \sqrt{4}\sqrt{2}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = \frac{2\left(2 \pm \sqrt{2}\right)}{2} = 2 \pm \sqrt{2}$$

$x = 2 + \sqrt{2}$ or $x = 2 - \sqrt{2}$

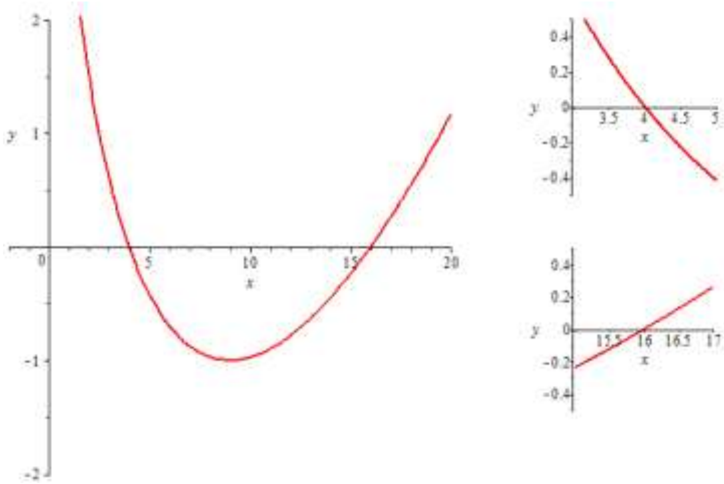
(3) $2x^2 - 3x = -2$
$2x^2 - 3x + 2 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(2)}}{2(2)} = \frac{3 \pm \sqrt{9-16}}{4} = \frac{4 \pm \sqrt{-7}}{2}$$

$$= No\ real\ solutions$$

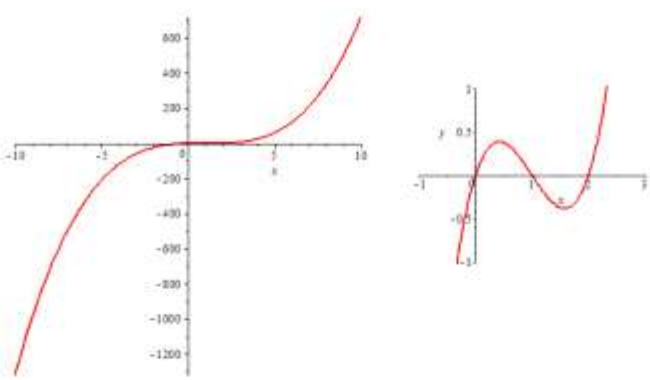
Ex.) Solve each equation – use a graphic method to solve if you get stuck

(1) $x - 6\sqrt{x} + 8 = 0$
Because the equation is not in the form $ax^2 + bx + c = 0$ the quadratic formula cannot be used. Instead, graph the equation and determine where the graph intersects the x-axis.



It appears the graph intersects the x-axis at $x = 4$ and $x = 16$. Zooming in (figures above on the right), one finds these are the exact values. Checking by substitution verifies each solution.

(2) $x^3 - 3x^2 + 2x = 0$
Again, the equation is not in the form $ax^2 + bx + c = 0$ so the quadratic formula cannot be used. Let's graph again.

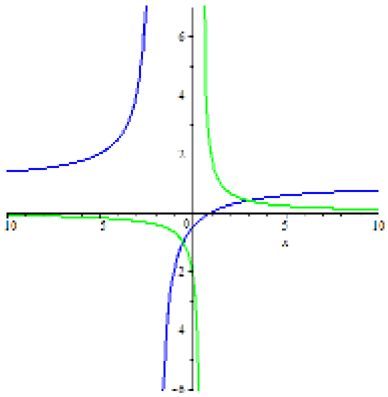


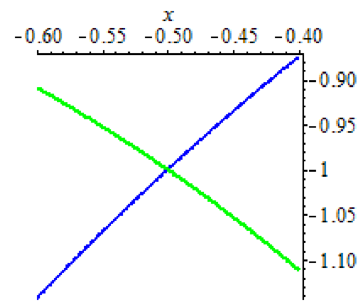
From the original picture it is rather difficult to determine the solution. Zooming in (figure above on the right), one finds these are the exact values at $x = 0, x = 1$ and $x = 2$. Checking by substitution verifies each solution.

(3) $\frac{x-1}{x+2} = \frac{2}{2x-1}$

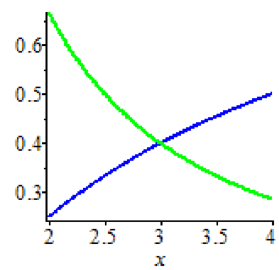
You can graph letting $y_1 = \frac{x-1}{x+2}$ and $y_2 = \frac{2}{2x-1}$.

Using a graphing tool, you can see there is a solution near $x = -1$ and $x = 3$.





Zooming in, one finds these are the exact values at $x = -0.5$ and $x = 3$. Checking by substitution verifies each solution.



This can be verified by solving the equation algebraically:

$$\frac{x-1}{x+2} = \frac{2}{2x-1} \Rightarrow (x-1)(2x-1) = 2(x+2) \Rightarrow 2x^2 - 3x + 1 = 2x + 4$$
$$\Rightarrow 2x^2 - 5x - 3 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(-3)}}{2(2)} = \frac{5 \pm \sqrt{25+24}}{4} = \frac{5 \pm \sqrt{49}}{4}$$

$$= \frac{5 \pm 7}{4} = \begin{cases} \frac{5+7}{4} = \frac{12}{4} = 3 \\ \text{or} \\ \frac{5-7}{4} = \frac{-2}{4} = -0.5 \end{cases}$$

4. Applications

Let's now readdress the original problem from the beginning. Recall, you were asked to find the air temperature when the beetle's oxygen consumption is 175 microliters per gram per hour given

$$C = 0.45t^2 - 1.65t + 50.75, \qquad 10 \le t \le 25.$$

Thus, the equation you must solve is

$$175 = 0.45t^2 - 1.65t + 50.75$$

This is in the form of a quadratic equation, therefore you may use the quadratic equation.

$$175 = 0.45t^2 - 1.65t + 50.75$$
$$0.45t^2 - 1.65t - 124.25 = 0$$
$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-1.65) \pm \sqrt{(-1.65)^2 - 4(0.45)(-124.25)}}{2(0.45)}$$
$$= \frac{1.65 \pm \sqrt{2.7225 + 223.65}}{0.9} = \frac{1.65 \pm \sqrt{226.3725}}{0.9}$$
$$= \begin{cases} \dfrac{1.65 + \sqrt{226.3725}}{0.9} \approx 18.6 \\ \text{or} \\ \dfrac{1.65 - \sqrt{226.3725}}{0.9} \approx -14.9 \end{cases}$$

The equation is only applicable to temperatures between 10℃ and 25℃, therefore your solution is approximately 18.6℃.

Ex.) The height above ground of a projectile (rock, cannon ball,…) is given by

$$h = 80 + 64\,t\ - 16t^2$$

where h is measured in feet and t is the number of seconds after the projectile is fired. After how many seconds will it hit the ground?

The height of the ball when it hits the ground is zero. Here the goal is to find t when $h = 0$.

$$0 = 80 + 64\,t\ - 16t^2$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(64) \pm \sqrt{(64)^2 - 4(-16)(80)}}{2(-16)} = \frac{-64 \pm \sqrt{4096 + 5120}}{-32}$$
$$= \frac{-64 \pm \sqrt{9216}}{-32}$$
$$= \frac{-64 \pm 96}{-32} = \begin{cases} \dfrac{-64 + 96}{-32} = \dfrac{32}{-32} = -1 \\ \text{or} \\ \dfrac{-64 - 96}{-32} = \dfrac{-160}{-32} = 5 \end{cases}$$

Because time cannot be negative, the ball hits the ground after 5 seconds.