ITEC122 2007fall-ibarland
Summations

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Rosen p.229–232 has a fine introduction to summations. The key element to remember is that $\sum$ is just a notation; whenever you see it you should mentally expand it into the sum it represents.

Here are a few problems from Rosen that introduce some standard tricks:

- Rosen 3.2, #15a:

$$\begin{aligned}\sum_{j=0}^{8} 3 \cdot 2^j &= 3 \cdot 2^0 + 3 \cdot 2^1 + 3 \cdot 2^2 + \cdots + 3 \cdot 2^8 \\ &= 3(2^0 + 2^1 + 2^2 + \cdots + 2^8) \\ &= 3 \cdot \sum_{j=0}^{8} 2^j \\ &= 3 \cdot (2^9 - 1) \text{by Rosen 3.2 Th'm 1}\end{aligned}$$

  The handy trick is that you can pull out the constant factor 3.

- Rosen 3.2, #17d: The previous trick is often useful in double sums:

$$\begin{aligned}\sum_{i=0}^{2}\sum_{j=0}^{3} ij &= \sum_{i=0}^{2}\left(\sum_{j=0}^{3} ij\right) \\ &= \sum_{i=0}^{2}\left(i\sum_{j=0}^{3} j\right) \\ &= \sum_{i=0}^{2}(i \cdot 6) \\ &= 6\sum_{i=0}^{2} i \\ &= 6 \cdot 3 \\ &= 18\end{aligned}$$

Why was it valid, in the first line, to factor out i from the inner sum? Because (with respect to the inner sum over j) it was a constant. Again, writing it out explicitly makes this clear.

- Difference of sums: When a sum's initial index isn't a nice even 0 or 1, often we can express the sum as a difference of two others. See Rosen Section 3.2, Example 15:

  $$\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2$$

  , and now each of the two sums can be individually computed from Section 3.2, Table 2.

- $\sum_{i=4}^{7}(2i-6) = 2+4+6+8 = 20$. This sum notation can also be written $\sum_{i\in\{4,5,6,7\}}$. or $\sum_{i\in[4,7]}$.

  Note: there's an easier summation notation for 2+4+6+8: If we let $j = i-3$, then when $i = 4$ then $j = 1$; and when $i = 7$ then $j = 4$; thus

  $$\sum_{i=4}^{7}(2i-6) = \sum_{j=1}^{4} 2j$$

  . (Or even start the sum from 0 :-)

- Know this sum cold:

  $$\sum_{i=0}^{n} i = n(n+1)/2$$

  . [the triangular numbers]

- Know this sum cold as well:

  $$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

  . [examples for n=2,3]

- This previous example generalizes:

  $$\sum_{i=0}^{n} b^i = (b^{n+1}-1)/(b-1)$$

  . [consider $b = 10 : 1 + 10 + 100 + 1000 = 1/9 * 9999$]

  When $b < 1$, we can take the infinite sum. In particular, $1 + 1/2 + 1/4 + +... = 2$. $1 + 1/10 + 1/100 + ... = 1.11111 = 10/9$. Note that $.9 + .9^2 + ...$ will also converge (to what?).

  Similarly: Is .432432432432... rational? $= 0.432{*}\sum_{i=0}^{\infty} 1/1000^i = 0.432(-1/(-999/1000)) =$ 432/999. Indeed, this generalizes: any repeating-decimal is rational.

- Now try: $\sum_{i=51}^{100} 2^i$. Split into two different sums; subtract. Actually, it's a bit moot for 2^i, because the first 50 terms altogether weren't as big as the 51st, and the 51-54rd are nearly sixteen times as big.

- Double sums:
  $\sum_i \sum_j i$
  $\sum_i \sum_j ij$
  $\sum_i \sum_j j^i$
  What if we switch order? To find out, expand!

- Consider expectations:
  the "mean" value of a fair six-sided die is $\sum i \cdot; (1/6)$. For an n-sided die, $\sum i/n$.
  How about for a weighted 6-sided die, where the two-pips has been artfully changed into a three-pips side:

  Expected number of tosses until a coin(die) comes up heads(6)? ... Can expand as rows and columns; arrange creatively and re-add. [Okay, it's a bit fishy w/ infinite series, but we'll hush that up.]