Critical Numbers of Rational Functions

Recall, to find a critical number, we first found the derivative of a function, then examined where the derivative was equal to zero and where the derivative was undefined. If the derivative of a function has both a numerator and a denominator, then determining where the derivative is equal to zero means to set the numerator equal to zero and solve the resulting equation. Similarly, to determine where the derivative is undefined, set the denominator equal to zero and solve the resulting equation.

The syntax to extract the numerator and denominator as separate expressions include

\begin{align*}
\text{\texttt{numer}(f)}; \quad &\text{#extracts the numerator of the expression } f \\
\text{\texttt{denom}(f)}; \quad &\text{#extracts the denominator of the expression } f
\end{align*}

Let's consider the function \( f(x) = \frac{x^2}{x^2 + 3} \). For the function, we will identify the critical numbers.

Begin by defining the function.

\begin{verbatim}
> f := x \mapsto \frac{x^2}{x^2 + 3};
\end{verbatim}

\[ f := x \mapsto \frac{x^2}{x^2 + 3} \quad (1.1) \]

Next, find the derivative.

\begin{verbatim}
> \text{\texttt{diff}(f(x), x)};
\end{verbatim}

\[ \frac{2x}{x^2 + 3} - \frac{2x^3}{(x^2 + 3)^2} \quad (1.2) \]

It appears a bit messy, so combine the fractions by simplifying the result. Let's also give it a name, say \( F1 \).

\begin{verbatim}
> F1 := \text{\texttt{simplify}}(\%);
\end{verbatim}

\[ F1 := \frac{6x}{(x^2 + 3)^2} \quad (1.3) \]

That looks a bit better. Our task is not to find the critical numbers. Let's extract the numerator and identify what will make the derivative equal to zero (i.e., what makes the numerator equal to zero).

\begin{verbatim}
> solve(\text{\texttt{numer}(F1)} = 0, x);
\end{verbatim}

\[ 0 \quad (1.4) \]
Therefore, $x = 0$ is one critical number. This time, we will extract the denominator and determine where the derivative is undefined (i.e., what makes the denominator equal to zero).

```plaintext
> solve(denom(F1) = 0, x);
```

\[ 1\sqrt{3}, -1\sqrt{3}, 1\sqrt{3}, -1\sqrt{3} \]  

(1.5)

As can be seen there are four complex solutions. Because we only desire real values, we may ignore these and conclude $x = 0$ is the only critical number.

**Example**

Find all critical numbers for $g(x) = \frac{x^2 - 3}{x - 2}$

**Solution**

Define the function.

```plaintext
> g := x -> \frac{x^2 - 3}{x - 2};
```

\[ g := x \rightarrow \frac{x^2 - 3}{x - 2} \]  

(1.1.1.1)

Find the derivative, simplifying the result.

```plaintext
> diff(g(x), x);
```

\[ \frac{2x}{x - 2} - \frac{x^2 - 3}{(x - 2)^2} \]  

(1.1.1.2)

```plaintext
> G1 := simplify(%);
```

\[ G1 := \frac{x^2 - 4x + 3}{(x - 2)^2} \]  

(1.1.1.3)

Now determine where the derivative equals zero and is undefined.

```plaintext
> solve(numer(G1) = 0, x);
```

\[ 3, 1 \]  

(1.1.1.4)

```plaintext
> solve(denom(G1) = 0, x);
```

\[ 2, 2 \]  

(1.1.1.5)

It appears that there are three critical numbers, but a closer examination of the values show that $x = 2$ is not a member of the domain of the original function $g(x)$, therefore is excluded as a critical number. Thus, the only critical numbers for this problem are $x = 1$ and $x = 3$. \[ \square \]
Critical Numbers

Using *numer* and *denom* is all well and good if the resulting derivative is a rational function; however, most functions we have investigated in class are not of this form. In fact, it may actually prove to be a simpler problem.

Let's consider the function \( f(x) = (x + 2)^2(x - 1) \). For the function, we will identify the critical numbers.

Begin by defining the function.

\[
> f := x \mapsto (x + 2)^2(x - 1);
\]

\[
f := x \mapsto (x + 2)^2(x - 1) \tag{2.1}
\]

Next, find the derivative.

\[
> \text{diff}(f(x), x);
\]

\[
2(x + 2)(x - 1) + (x + 2)^2 \tag{2.2}
\]

It may be possible to simplify the result. Let's also give it a name, say \( F1 \).

\[
> F1 := \text{simplify}(\%);
\]

\[
F1 := 3x^2 + 6x \tag{2.3}
\]

That looks a bit better. Our task is not to find the critical numbers. Because the result is not a rational expression, we can just determine where it is equal to zero and where it is undefined.

\[
> \text{solve}(F1 = 0, x);
\]

\[
-2, 0 \tag{2.4}
\]

Therefore, \( x = -2 \) and \( x = 0 \) are two critical number. The derivative is defined for all values of \( x \), thus there is no place where the derivative is undefined. Therefore, the only two critical numbers are \( x = -2 \) and \( x = 0 \).

**CriticalPoints as a Maple command**

Maple has a command that can provide the critical points of a function which is much faster than the methods investigated earlier. The command is a part of the *Student[Calculus1]* package, so you will need to load it first. A colon is used to supress the output, but if you wish to see all of the commands available in the package, use a semi-colon.

\[
> \text{with(Student[Calculus1])}:
\]

The syntax used in the command is

\[
\text{CriticalPoints}(f(x), x);
\]

The result is a list of all critical points of the function \( f(x) \). In the event the independent variable can be determined from the expression, the parameter \( x \) does not need to be included in the command sequence. If a closed interval \([a, b]\) is specified, then you may use

\[
\text{CriticalPoints}(f(x), x=a..b);
\]
to find all the critical points on that interval.

Let's find the critical points of \( f(x) = (3 - x)e^{x - 3} \).

\[
> \text{CriticalPoints}(3 - x) \cdot \exp(x - 3));
\]

\[
\text{[2]}
\]

As can be seen there is only one critical point, \( x = 2 \).

Let's try a few more. Find the critical points of \( h(x) = x^2 \cdot 3 \sqrt{1 - 2x} + 1 \) and \( v(t) = 6 \sin \left( \frac{t}{2} \right) \).

\[
> \text{CriticalPoints}(x^2 \cdot (1 - 2 \cdot x)^{\frac{1}{3}} + 1);
\]

\[
[0, \frac{3}{7}, \frac{1}{2}]
\]

\[
> \text{CriticalPoints}(6 \cdot \sin \left( \frac{t}{2} \right));
\]

Warning, the expression has an infinite number of critical points, some examples of which are given

\[
[\pi, 3 \pi]
\]

As can be seen there are three critical points for \( h(x) \), but for \( v(t) \) there is an infinite number of critical points - hence the warning. If \( v(t) \) is limited to values of \( t \) on the interval \( \left[ -\frac{7 \pi}{2}, \frac{5 \pi}{3} \right] \), we can find the critical points on that interval.

\[
> \text{CriticalPoints}(6 \cdot \sin \left( \frac{t}{2} \right), t=\frac{7 \cdot \pi}{2}, \frac{5 \cdot \pi}{3});
\]

\[
[-3 \pi, -\pi, \pi]
\]

Thus, \( t = -3 \pi, t = -\pi, \) and \( t = \pi \) are all critical numbers for \( v(t) \) on \( \left[ -\frac{7 \pi}{2}, \frac{5 \pi}{3} \right] \).

\[\text{Maximums & Minimums}\]

Once critical numbers are determined, one can attempt to identify whether a function has a maximum or minimum at each critical point and what is the actual maximum or minimum value.

Suppose our function is \( f(x) = \frac{2x^5 - 7x^4 - 30x^3 + 43x^2 + 148x + 60}{x^2 + 4}, \quad -4 \leq x \leq 5 \). Initially, let's use a graph to help us determine if there exists a minimum (bottom of a valley) or maximum (top of a hill).
\[ f(x) = \frac{2x^5 - 7x^4 - 30x^3 + 43x^2 + 148x + 60}{x^2 + 4} \]

(3.1)

As can be seen there are hills and valleys, as well as a starting and ending point for the graph (it is on a closed interval afterall). Next, find all the critical numbers.

\[
> \text{plot}(f(x), x = -4 \ldots 5, \text{color}=\text{blue}, \text{thickness}=3) ;
\]

As can be seen there are hills and valleys, as well as a starting and ending point for the graph (it is on a closed interval afterall). Next, find all the critical numbers.

\[
> \text{CriticalPoints}(f(x), x = -4 \ldots 5) ; \\
[ -2 ]
\]

That is interesting. According to Maple, there is only one critical point; however, it is clear from the graph that there should be more. One additional part of the \texttt{CriticalPoints} command structure is to include \texttt{numeric=true} in order to provide floating-point computations in the results (i.e., decimal solutions).
> CriticalPoints\( f(x), x = -4 .5, numeric = true \);  
\[
\begin{array}{l}
[ -2.000000000, -1.002639617, 1.157685332, 4.075686459 ]
\end{array}
\]  
(3.3)

As expected from the graph, there should be a total of four critical points on the interval and we now have all four. Based upon the graph, there exists a relative maximum at \( x = -2 \), a relative minimum at \( x = -1.002639617 \), an absolute maximum at \( x = -1.157685332 \), and a relative minimum at \( x = 4.075685459 \). Where is the absolute minimum?

Recall on a closed interval, by the Extreme Value Theorem, a continuous function has both a minimum and maximum on the interval. Because we are focused on a closed interval, we should also include the endpoints in our investigation. When \( x = -4 \) the function achieves its lowest values, or in other words is at its minimum.

To determine the values of the minimums (relative and absolute) and and maximums (relative and absolute), first identify where they occur. Granted, we did this when identifiying the critical points. Note, the numbers \( x = -4 \) and \( x = -2 \) are easy to use, but note the other critical points are approximations.

> AbsMin := f(-4);  
\[
AbsMin := \frac{-441}{5}
\]  
(3.4)

> RelMax1 := f(-2);  
\[
RelMax1 := 0
\]  
(3.5)

> RelMin1 := f(-1.002639617);  
\[
RelMin1 := -4.800105489
\]  
(3.6)

> AbsMax := f(1.157685332);  
\[
AbsMax := 43.81937035
\]  
(3.7)

> RelMin2 := f(4.075685459);  
\[
RelMin2 := -16.29589718
\]  
(3.8)

If our goal is simply to find the absolute maximum and absolute minimum of a function over a given interval, we could use the maximize and minimize commands.

> maximize\( (f(x), x = -4 .5) \);  
\[
\begin{array}{l}
(2 \text{ RootOf}(3 _Z^2 - 13 _Z^4 + 31 _Z^3 - 118 _Z^2 - 18 _Z + 148, 1.157685332) )^5 \\
- 7 \text{ RootOf}(3 _Z^2 - 13 _Z^4 + 31 _Z^3 - 118 _Z^2 - 18 _Z + 148, 1.157685332) )^4 \\
- 30 \text{ RootOf}(3 _Z^2 - 13 _Z^4 + 31 _Z^3 - 118 _Z^2 - 18 _Z + 148, 1.157685332) )^3 \\
+ 43 \text{ RootOf}(3 _Z^2 - 13 _Z^4 + 31 _Z^3 - 118 _Z^2 - 18 _Z + 148, 1.157685332) )^2 \\
+ 148 \text{ RootOf}(3 _Z^2 - 13 _Z^4 + 31 _Z^3 - 118 _Z^2 - 18 _Z + 148, 1.157685332) ) \\
+ 60 ) / \left( \text{ RootOf}(3 _Z^2 - 13 _Z^4 + 31 _Z^3 - 118 _Z^2 - 18 _Z + 148, 1.157685332) )^2 \\
+ 4 )
\end{array}
\]  
(3.9)

That appears a bit messy, so let's evaluate the expression.

> evalf\( (\% ) \);  
\[
43.81937035
\]  
(3.10)
Increasing & Decreasing Functions

Recall, a function \( f(x) \) is increasing on an interval \((a, b)\) provided \( \frac{df}{dx} > 0 \) for all \( x \in (a, b) \).

Similarly, a function \( f(x) \) is decreasing on an interval \((a, b)\) provided \( \frac{df}{dx} < 0 \) for all \( x \in (a, b) \).

Now we know how to find the critical points, we can subdivide the real number line into intervals and use the definitions to determine where a function is increasing and decreasing.

Example - Numerical Solution

Determine the open intervals where the function \( h(t) = (t + 2)^2(t - 1) \) is increasing or decreasing.

Solution

Begin by defining the function.

\[
> h := t \mapsto (t + 2)^2(t - 1); \quad h := t \mapsto (t + 2)^2(t - 1)
\]  

Next, determine the critical points. Recall that the package \texttt{Student[Calculus1]} needs to have been loaded prior to using the \texttt{CriticalPoints} command.

\[
> \text{CriticalPoints}(h(t)); \quad [-2, 0]
\]

This implies the real number line is subdivided into intervals \(( -\infty, -2) \cup (-2, 0) \cup (0, \infty) \). Now, select a value within each interval and substitute it into the derivative. If the result is positive, the function is increasing on that interval \( \left( \frac{dh}{dt} > 0 \right) \). On the other hand, if the result is negative, the function is decreasing on that interval \( \left( \frac{dh}{dt} < 0 \right) \).

\[
> \textbf{D}(h)\ (-4); \quad \text{#substituting a number from the interval } (-\infty,-2) \text{ into the derivative} \quad 24
\]

\[
> \textbf{D}(h)\ (-1); \quad \text{#substituting a number from the interval } (-2, 0) \text{ into the derivative} \quad (4.1.1.4)
\]
Therefore, the function $h(t)$ is increasing on $(-\infty, -2) \cup (0, \infty)$ and is decreasing on $(-2, 0)$.

**Example - Graphical Solution**

Determine the open intervals where the function $k(x) = 3x^4 - 22x^3 - 18x^2 + 6$ is increasing or decreasing using a graph.

**Solution**

Begin by defining the function.

```plaintext
> k := x->3*x^4 - 22*x^3 - 18*x^2 + 6;

$\quad k := x \mapsto 3x^4 - 22x^3 - 18x^2 + 6$  \hspace{1cm} (4.2.1.1)
```

Next, determine the critical points. Recall that the package `Student(Calculus1)` needs to have been loaded prior to using the `CriticalPoints` command.

```plaintext
> CriticalPoints(k(x));

$\quad \left[-\frac{1}{2}, 0, 6\right]$  \hspace{1cm} (4.2.1.2)
```

This implies the real number line is subdivided into intervals

$(-\infty, -\frac{1}{2}) \cup \left(-\frac{1}{2}, 0\right) \cup (0, 6) \cup (6, \infty)$. Now, let's look at the graph of the derivative.

```plaintext
> plot(diff(k(x), x), x = -5..8, color = red);
```
Remember that we are looking at the graph of the derivative. From the graph it is clear that on the intervals \(( -\infty, -\frac{1}{2} \) and \(( 0, 6)\) the derivative is below the x-axis, thus \(\frac{dk}{dx} < 0\) on those intervals. Therefore, the function \(h(t)\) is **decreasing** on \(( -\infty, -\frac{1}{2} \) \(\cup\) \(( 0, 6)\). Similarly, it is clear that the derivative is above the x-axis on \(( 6, \infty)\). But what about the interval \(( -\frac{1}{2}, 0)\)?

To get a better view, zoom into the area in question.

\[
\text{plot}(\text{diff}(k(x), x), x=-1..1, \text{color} = \text{red});
\]
As can now be seen clearer, the derivative is above the x-axis on the interval \((-\frac{1}{2}, 0)\) and therefore is positive on the interval. Thus the function is \textit{increasing} on \((-\frac{1}{2}, 0) \cup (6, \infty)\).