A Probability Problem

1 Maximizing a Probability Function

This problem is extracted from the book *Exploring Mathematics with Scientific Notebook* co-authored by Wei-Chi Yang and Jonathan Lewin.

In this example, we suppose that we have n white balls and n black balls which we are going to place in two urns A and B in any way we please, as long as at least one ball is placed into each urn. After this has been done, a second person walks into the room and selects one ball at random. Our problem is to maximize the probability that this person draws a white ball.

We suppose that the distribution of the balls in the urns A and B is as described in the following table:

	A	B
Number of White Balls	x	$n-x$
Number of Black Balls	y	$n-y$

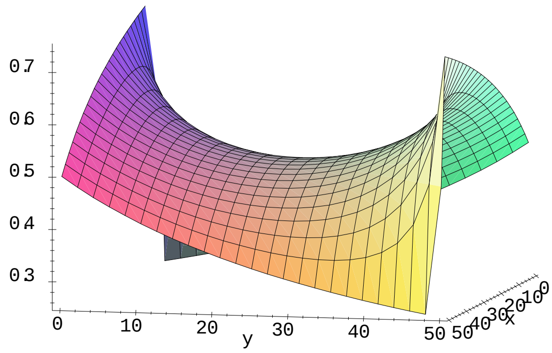
If $P(x, y, n)$ is the probability that a single ball drawn at random will be white then

$$P(x,y,n) = \frac{1}{2}\left(\frac{x}{x+y} + \frac{n-x}{2n-x-y}\right).$$

From now on we shall assume that $n = 50$. We begin our study of the function by looking at the following table which shows the values of $P(x, y, 50)$ at a few selected points (x, y).

$P(0,1,50) = .25253$	$P(1,0,50) = .74747$
$P(1,1,50) = .5$	$P(2,1,50) = .58076$
$P(1,2,50) = .41924$	$P(25,25,50) = .5$
$P(50,1,50) = .4902$	$P(1,50,50) = .5098$
$P(50,0,50) = .4902$	$P(50,49,50) = .25253$
$P(49,50,50) = .74747$	$P(49,49,50) = .5$

To solve the problem we need to find the maximum value of the expression $P(x, y, 50)$ as the point (x, y) varies through the rectangle $[0, 50] \times [0, 50]$ from which the points $(0, 0)$ and $(50, 50)$ have been removed. If we sketch the graph $z = P(x, y, 50)$ then we obtain the following surface:



From the looks of this surface it seems unlikely that the maximum value of z will be achieved at a critical point. The maximum appears to be at the left or right extremities of the figure. As a matter of fact, if we point at the equations

$$\begin{aligned}\frac{\partial}{\partial x}P(x,y,50) &= 0\\ \frac{\partial}{\partial y}P(x,y,50) &= 0\end{aligned}$$

and click on Solve and Exact then we obtain

$$\{y = 25, x = 25\}.$$

As we have already seen, the maximum value of z does not occur at the point $(25, 25)$.

We now examine the boundary behavior of the function. There are four cases to consider

The Case $x = 0$ and $1 \le y \le 50$ We define $g(y) = P(0, y, 50)$ for $1 \le y \le 50$. Point at this definition of $g(y)$ and click on Define and New Definition. Since

$$g(y) = \frac{25}{100 - y}$$

for each y we see that the maximum value of $g(y)$ is $g(50) = \frac{1}{2}$.

The Case $y = 0$ and $1 \le x \le 50$ We define $g(x) = P(x, 0, 50)$ for $1 \le x \le 50$. Point at this definition of $g(y)$ and click on Define and New Definition. Since

$$g(x) = \frac{1}{2} + \frac{1}{2}\frac{50 - x}{100 - x} = 1 - \frac{25}{100 - x}$$

for each x, we see that the maximum value of this function is $g(1) = .74747$.

The Case $x = 50$ and $0 \le y \le 49$ We define $g(y) = P(50, y, 50)$ for $0 \le y \le 49$. Point at this definition of $g(y)$ and click on Define and New Definition. Since

$$g(y) = \frac{25}{50 + y}$$

for each y, we see that the maximum value of this function is $g(0) = .5$.

The Case $y = 50$ **and** $0 \leq x \leq 49$ We define $g(x) = P(x, 50, 50)$ for $0 \leq x \leq 49$. Point at this definition of $g(y)$ and click on Define and New Definition. Since

$$g(x) = 1 - \frac{50}{x + 50}$$

for each x, we see that the maximum value of this function is $g(49) = .74747$.

2 Conclusion

We conclude that the expression $P(x, y, 50)$ takes a maximum value of .74747 at the point $(0, 1)$ and again at the point $(49, 50)$. This means that we can maximize the probability that a white ball will be selected by placing no white ball and just one black ball in urn A and all the other balls in urn B. Alternatively we can place no white ball and just one black ball in urn B and all the other balls in urn A.

2.1 Some Variations of the Probability Problem

Some of the variations suggested here may be suitable for presentation in the classroom. Others may be suitable for student projects.

1. Repeat the preceding probability problem assuming that the selection of the ball will be made in such a way that the probability that the selection will be made from urn A is $\frac{1}{3}$ and the probability that the selection will be made from urn B is $\frac{2}{3}$.

2. Extend the preceding variation to the general case in which the probability of selecting a ball from urn A is some number γ satisfying $0 < \gamma < 1$ and the probability of selecting the ball from urn B is $1 - \gamma$.

3. Investigate the problem of determining how the balls should be placed in order to minimize the probability that the selected ball be white. Of course, this is simply the problem of maximizing the probability that a black ball be selected.

4. Suppose that the selection of the ball results in payoffs as described in the following table

	From urn A	From urn B
white ball	α_A	α_B
black ball	β_A	β_B

   Study the payoffs that result from different placement of the balls.

5. Determine the maximum value of the expression $P(x, k, n)$ where k is a given integer satisfying $1 \leq k \leq n - 1$. Find the value x_k of x at which this maximum occurs. You will find that

$$x_k = \frac{\sqrt{k}(2n - k) - k\sqrt{n - k}}{\sqrt{n - k} + \sqrt{k}}$$

   and that the maximum value of $P(x, k, n)$ is

$$P(x_k, k, n) = \frac{3n - 2\sqrt{(k(n - k))}}{4n}$$