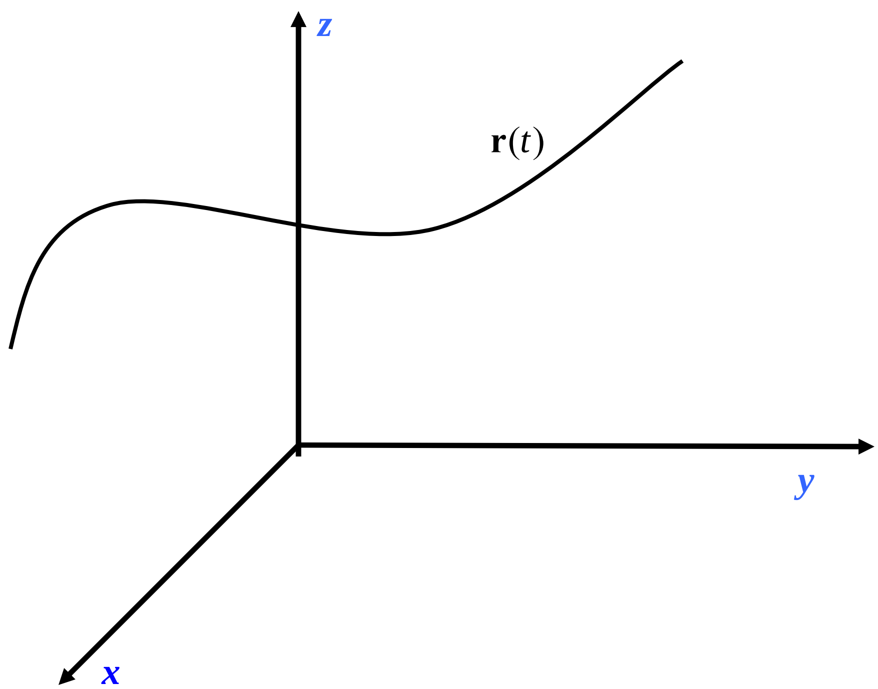
Section 10.3: Arc Length and Curvature

Practice HW from Stewart Textbook (not to hand in)
p. 714 # 1-4 odd, 11-19

Arc Length

Suppose we want to calculate the length of vector function $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ over some interval $a \le t \le b$.



We determine the arc length using the following formula.

Formula for Arc Length

For a vector valued function $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, the arc length L of the curve for the closed interval from $t = a$ to $t = b$ is given by the formula

$$L = \int_a^b |\mathbf{r}'(t)|\, dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$

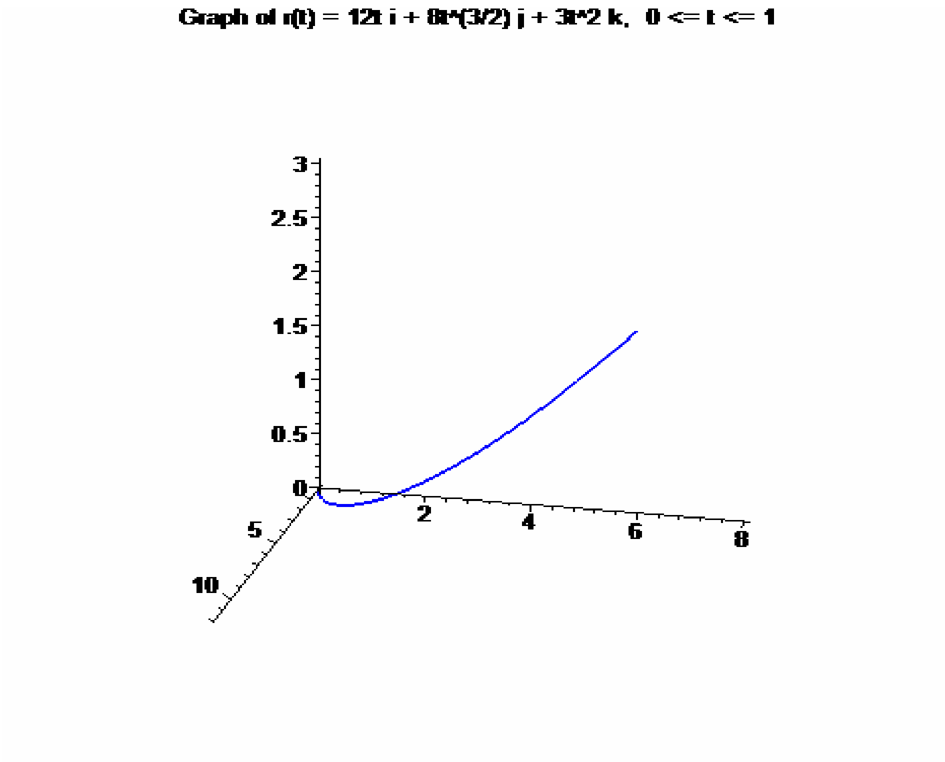
Example 1: Find the length of the curve $\mathbf{r}(t) = 12t\,\mathbf{i} + 8t^{3/2}\,\mathbf{j} + 3t^2\,\mathbf{k},\ \ 0 \le t \le 1$.

.

Solution:

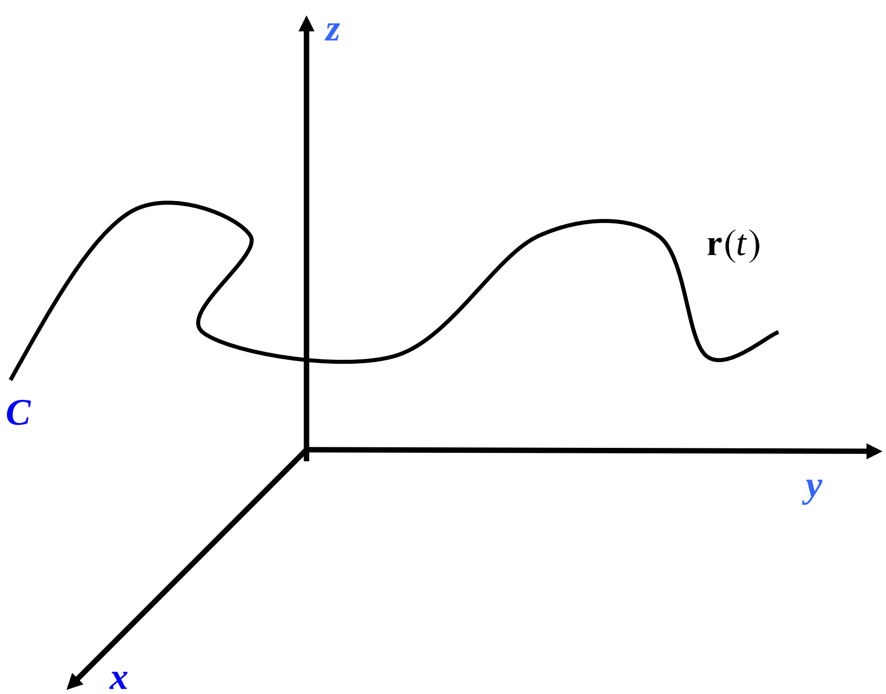
$\blacksquare$

Note: The following graph represents a sketch of the vector valued function
$\mathbf{r}(t) = 12t\,\mathbf{i} + 8t^{3/2}\,\mathbf{j} + 3t^2\,\mathbf{k},\ \ 0 \le t \le 1$



Curvature

Suppose we are given the graph of the vector function $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ given by the curve C.



Recall that the formula for the unit tangent vector is given by

$$\boldsymbol{T}(\mathrm{t}) = \frac{\boldsymbol{r}'(\mathrm{t})}{|\boldsymbol{r}'(\mathrm{t})|} = \frac{1}{|\boldsymbol{r}'(\mathrm{t})|}\boldsymbol{r}'(\mathrm{t})$$

The unit tangent vector indicates the direction of the curve. We would like to be able to measure how fast the curve changes, which corresponds to how fast the direction of the unit tangent vector changes. The *curvature* measures how quickly the curve changes direction at a specific point. It can be shown to be found by the following formula.

Formula for Curvature

For a vector valued function $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ with unit tangent vector **T**, the *curvature* κ is given by the formula

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

Note: It can be shown that an alternative (and often) easier method for finding the curvature is given by the formula

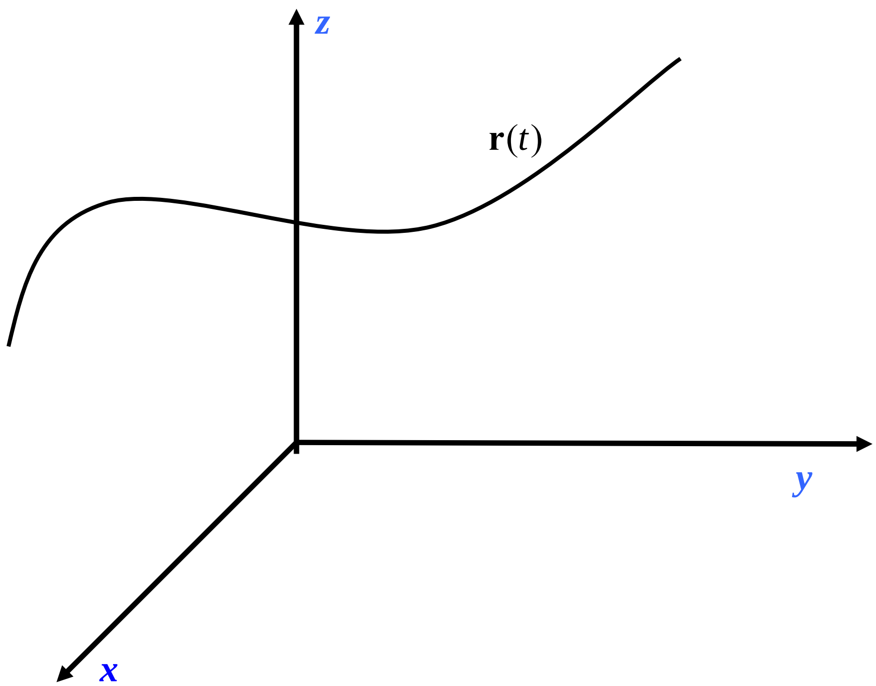
$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Example 2: Find the curvature of the vector function $\mathbf{r}(t) = < t, 5\sin t, 5\cos t >$. Use your result to find the curvature at the point $(\pi, 0, -5)$

Solution:

Calculating the Unit Normal Vector

In describing motion, sometimes it can be an advantage to calculate a vector orthogonal to the unit tangent vector **T** of the vector function $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$.



One method is the use the fact that the **T** is a unit vector. Then, we know that

$$\mathbf{T}(t) \cdot \mathbf{T}(t) = |\mathbf{T}(t)|^2 = 1$$

If we use the product rule and differentiate both sides of this equation, we obtain

$$\mathbf{T}(t) \cdot \mathbf{T}'(t) + \mathbf{T}'(t) \cdot \mathbf{T}(t) = 0$$

or

$$2\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0.$$

Thus,

$$\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0$$

Hence, the derivative of **T**, $\mathbf{T}'$, is orthogonal to **T**. Normalizing **T** will give the principal unit normal vector **N** at *t*.

Principal Unit Normal Vector N

For the vector valued function $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ with unit tangent vector **T**, the unit normal vector, assuming $\mathbf{T}'(t) \neq 0$, is given by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\,\mathbf{T}'(t)\,|} = \frac{1}{|\,\mathbf{T}'(t)\,|}\mathbf{T}'(t)$$

Note: Unless the function $\mathbf{r}(t)$ is simple, the unit normal vector **N** can be very computationally intensive. Maple can be a very convenient tool for simplify the computations.

Example 3: Find the unit normal vector **N** for the vector $\mathbf{r}(t) = < t, 5\sin t, 5\cos t >$. Use your result to find the unit normal vector at the point $(\pi, 0, -5)$

Solution: To find the unit normal vector **N**, we must first find the unit tangent vector **T**. Recall that the formula for the unit tangent vector **T** is given by

$$\mathbf{T(t)} = \frac{1}{|\,\mathbf{r}'(t)\,|}\mathbf{r}'(t)$$

For $\mathbf{r}(t) = < t, 5\sin t, 5\cos t >$, we have $\mathbf{r}'(t) = < 1, 5\cos t, -5\sin t >$ and

$$|\,\mathbf{r}'(t)\,| = \sqrt{(1)^2 + (5\cos t)^2 + (-5\sin t)^2} = \sqrt{1 + 25\cos^2 t + 25\sin^2 t}$$
$$= \sqrt{1 + 25(\cos^2 t + \sin^2 t)} = \sqrt{1 + 25(1)} = \sqrt{26}$$

Thus,

$$\mathbf{T(t)} = \frac{1}{|\,\mathbf{r}'(t)\,|}\mathbf{r}'(t) = \frac{1}{\sqrt{26}} < 1, 5\cos t, -5\sin t > = < \frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\cos t, -\frac{5}{\sqrt{26}}\sin t >$$

To get the unit normal vector, we use the formula

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\,\mathbf{T}'(t)\,|} = \frac{1}{|\,\mathbf{T}'(t)\,|}\mathbf{T}'(t)$$

Using $\mathbf{T}(t) = < \frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\cos t, -\frac{5}{\sqrt{26}}\sin t >$, we see that $\mathbf{T}'(t) = < 0, -\frac{5}{\sqrt{26}}\sin t, -\frac{5}{\sqrt{26}}\cos t >$

and **(continued on next page)**

$$|\mathbf{T}'(t)| = \sqrt{(0)^2 + \left(-\frac{5}{\sqrt{26}}\sin t\right)^2 + \left(-\frac{5}{\sqrt{26}}\cos t\right)^2} = \sqrt{0 + \frac{25}{26}\sin^2 t + \frac{25}{26}\cos^2 t}$$

$$= \sqrt{\frac{25}{26}(\sin^2 t + \cos^2 t)} = \sqrt{\frac{25}{26}(1)} = \frac{5}{\sqrt{26}}$$

Then,

$$\mathbf{N}(t) = \frac{1}{|\mathbf{T}'(t)|}\mathbf{T}'(t) = \frac{\frac{1}{1}}{\frac{5}{\sqrt{26}}} < 0, -\frac{5}{\sqrt{26}}\sin t, -\frac{5}{\sqrt{26}}\cos t >$$

$$= \frac{\sqrt{26}}{5} < 0, -\frac{5}{\sqrt{26}}\sin t, -\frac{5}{\sqrt{26}}\cos t >$$

$$=< \frac{\sqrt{26}}{5}\cdot 0, \frac{\cancel{\sqrt{26}}}{\cancel{5}}\cdot\left(-\frac{\cancel{5}}{\cancel{\sqrt{26}}}\right)\sin t, \frac{\cancel{\sqrt{26}}}{\cancel{5}}\cdot\left(-\frac{\cancel{5}}{\cancel{\sqrt{26}}}\right)\cos t >$$

$$\boxed{=< 0, -\sin t, -\cos t >}$$

We last want to evaluate $\mathbf{N}(t) =< 0, -\sin t, -\cos t >$ at the point $(\pi, 0, -5)$. We need to find the value of t that produces this point. Since $\mathbf{r}(t) =< t, 5\sin t, 5\cos t >$, $t = \pi$ since and $5\sin t = 5\sin\pi = 5(0) = 0$ and $5\cos t = 5\cos\pi = 5(-1) = -5$. Thus,

$$\boxed{\begin{array}{ll} \textit{Normal Vector at Point} & \\ \quad (\pi, 0, -5) & = \mathbf{N}(\pi) =< 0, -\sin\pi, -\cos\pi >=< 0, -0, --1 >=< 0, 0, 1 > \\ \quad t = \pi & \end{array}}$$

The following shows a framed graph of $\mathbf{r}(t) =< t, 5\sin t, 5\cos t >$ (in green) with the unit tangent vector $\mathbf{T}(t) =< \frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\cos t, -\frac{5}{\sqrt{26}}\sin t >$ (in red) and unit normal vector $\mathbf{N}(t) =< 0, -\sin t, -\cos t >$ (in blue) displayed at the point $(\pi, 0, -5)$.