Logarithms

1. Common Logarithm

The term “logarithm” comes from combining the two Greek terms *logos* (“to calculate”) and *arithmos* (“a number”). Developed by John Napier in 1614, the purpose of a logarithm is to do multiplication and division by basic addition and subtraction.

Suppose you have a number x but wish to rewrite it as a power of 10. Some values are easy:

$$10{,}000 = 10^4, \frac{1}{10} = 10^{-1}, 1 = 10^0, etc.$$

However, most numbers are not easily expressed as a power of 10.

$$4 = 10^?, \qquad \frac{2}{3} = 10^?, \qquad 12 = 10^?$$

In general we want to find

$$x = 10^y$$

We call the number y the ***common logarithm*** of x which is written as

$$y = \log x.$$

In general, logarithms are expressed with a base, say base b. For the following equation

$$y = \log_b x$$

we say “y is the logarithm base b of x.” If no base is indicated, then it is assumed to be 10 and we are using the common logarithm.

Ex.) Simplify each

(i) $\log 100$
Because $100 = 10^2$, $\log 100 = \log 10^2 = 2$

(ii) $\log 0.001$
Because $0.001 = 10^{-3}$, $\log 0.001 = \log 10^{-3} = 3$

(iii) $\log \sqrt[3]{100}$
Because $\sqrt[3]{100} = 100^{\frac{1}{3}} = 10^{\frac{2}{3}}$, $\log \sqrt[3]{100} = \log 10^{\frac{2}{3}} = \frac{2}{3}$

(iv) $\log 1$
Because $1 = 10^0$, $\log 1 = \log 10^0 = 1$

In the last few examples, we were able to rewrite the argument of the logarithm as a power of 10. This is not always the case; however, you can use the relationship between common logarithms and exponential terms (base 10):

$$y = \log x \Leftrightarrow 10^y = x$$

Ex.) Simplify each

(i) $10^t = 25$
$\log 10^t = \log 25$
$t = \log 25 \approx 1.3979$

(ii) $10^a = 0.000064$
$\log 10^a = \log 0.000064$
$a = \log 0.000064 \approx -4.1938$

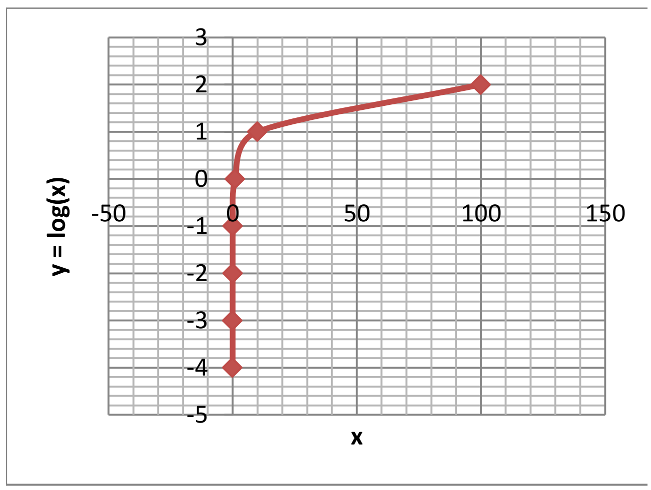
(iii) $\log x = 3.7$
$10^{\log x} = 10^{3.7}$
$x = 10^{3.7} \approx 5011.9$

(iv) $\log x = -5.5$
$10^{\log x} = 10^{-5.5}$
$x = 10^{-5.5} \approx 0.000003162$

Investigating the function $y = \log x$ requires examining at the values of x. From the previous examples, if x was between $x = 0$ and $x = 1$, the value of its logarithm was negative (less than zero) and if x was greater than 1, the value of the logarithm was positive (greater than zero).

Let's construct a table of values, selecting powers of 10 for x for simplicity and solve for y.

x	$y = \log x$
$0.0001 = 10^{-4}$	-4
$0.001 = 10^{-3}$	-3
$0.01 = 10^{-2}$	-2
$0.1 = 10^{-1}$	-1
$1 = 10^0$	0
$10 = 10^1$	1
$100 = 10^2$	2



As can be seen the graph does not appear to cross the y-axis. As a general rule, if $y = \log x$ then

$$x > 0.$$

In other words, taking the logarithm of a negative or zero is not possible!

2. Properties of Logarithms

Let's use a few examples to generate the general properties of logarithms (feel free to verify each on the calculator).

(1) $10^{\log 2} = 10^{0.301} = 2$

(2) $\log 1000 = \log 10^3 = 3$

(3) $\log 3 + \log 5 = \log 15 = 1.176$

(4) $\log 4000 = \log 1000 + \log 4 = 3 + \log 4 = 3.602$

(5) $\log 8 = \log 2^3 = 3 \log 2 = 0.903$

(6) $\log 0.2 = \log \frac{1}{5} = -\log 5 = -0.699$

(7) $\log \sqrt{5} = \log 5^{1/2} = \frac{1}{2} \log 5 = 0.349$

Properties

- $a = 10^{\log a}$
- $a = \log 10^a$
- $\log ab = \log a + \log b$
- $\log a^n = n \log a$

Ex.) Solve each equation.

(i)
$10^x = 16$
$\log 10^x = \log 16$
$x \log 10 = \log 16$
$x(1) = \log 16$
$x = \log 16$
This is the exact solution. For an estimated solution, refer to the calculator (or logarithm table). $\log 16 \approx 1.204$

(ii)
$5^x = 16$
$\log 5^x = \log 16$
$x \log 5 = \log 16$
$$x = \frac{\log 16}{\log 5} \approx 1.723$$

(iii) $3^{2x+1} = 100$

$(2x+1)\log 3 = 2$

$2x + 1 = \dfrac{2}{\log 3}$

$2x = \dfrac{2}{\log 3} - 1$

$x = \dfrac{1}{2}\left(\dfrac{2}{\log 3} - 1\right) \approx$

3. Applications

A. *pH scale*

"***pH***" means power of hydrogen. pH is a logarithmic scaled based on powers of 10 ansd is a measure of the concentration of hydrogen ions, H^+, in a liquid. By definition

$$pH = -\log[H^+]$$

This equation can be rewritten to determine the molar concentration, or the number of hydrogen ions, and is expressed as

$$10^{-pH} = [H^+]$$

Ex.) What is the pH of water if the at $25℃$ distilled water has molar concentration $[H^+] = \frac{10^{-7} moles\ of\ H^+}{Liter} = 10^{-7}M$?

$pH = -\log[H^+] = -\log[10^{-7}] = -(-7) = 7$

Thus, $pH(water) = 7$.

Ex.) What is the pH of blood given the hydrogen concentration is $4 \cdot 10^{-8}\ M$?

$$pH = -\log[H^+] = -\log[4 \cdot 10^{-8}] = -(\log[4] + \log[10^{-8}]) = -(\log[4] - 8)$$
$$\approx -(0.6 - 8) = -(-7.4) = 7.4$$

Thus, $pH(blood) \approx 7.4$.

Ex.) Coffee has a pH of 5.5. What is its hydrogen concentration?

$$[H^+] = 10^{-pH}\ M = 10^{-5.5}\ M = 10^{0.5} \cdot 10^{-6}\ M = \sqrt{10} \cdot 10^{-6}\ M$$
$$\approx 3.162 \cdot 10^{-6}\ M = 0.000003162\ M$$

The hydrogen concentration of coffee is approximately $0.000003162\ M$.

Thus, $pH(blood) \approx 7.4$.

Ex.) The pH of beer is 4.0. Is the hydrogen concentration higher or lower than that of water? By what factor?

$[H^+] = 10^{-pH}\ M = 10^{-4}\ M = 0.0001\ M$

The hydrogen concentration of water is $10^{-7}\ M$ (from the earlier example).

$$10^{-4} > 10^{-7}$$

Beer has a higher hydrogen concentration by a factor of $\frac{10^{-4}}{10^{-7}} = 10^{-4-(-7)} = 10^3$.

B. *Earthquake measurement*

For quite some time, the ***Richter Scale*** was used to measure the magnitude of an earthquake. The scale is a base 10 logarithmic scale. An earthquake measuring 5.0 on the Richter Scale has a shaking amplitude 10 times larger than one measuring 4.0 and corresponds to a 31.6 times larger release of energy.

In the 1970's the ***momentum magnitude scale*** replaced the Richter Scale (in most countries). A ***moment*** to seismologists is the product of the rock strength, the area of the fault between the two blocks, and the average amount of movement or slip of one block past the other. The moment magnitude uses the logarithm of the moment, calculated as

$$M_w = \log\left(\frac{m^{2/3}}{10^{10.7}}\right) = \log\left(m^{2/3}\right) - \log(10^{10.7}) = \log\left(m^{2/3}\right) - 10.7$$

In the above equation, m represents the moment measured in dyne-cm (or erg).

Ex.) The Tohaku earthquake (Fukushima, March 11, 2011) had moment magnitude $M_w = 9.0$. How much energy was released?

$M_w = \log\left(m^{2/3}\right) - 10.7$

$9 = \log\left(m^{2/3}\right) - 10.7$
$19.7 = \log\left(m^{2/3}\right)$
$10^{19.7} = 10^{\log\left(m^{2/3}\right)}$
$10^{19.7} = m^{2/3}$
$(10^{19.7})^{3/2} = \left(m^{2/3}\right)^{3/2}$
$3.548 \cdot 10^{29} \approx m$

Approximately $3.548 \cdot 10^{29}$ dyne-cm was released during the earthquake.

Ex.) What is the moment magnitude M_w when the moment is $5.0 \cdot 10^{26}$ dyne-cm?

$M_w = \log\left(m^{2/3}\right) - 10.7 = \log\left((5.0 \cdot 10^{26})^{2/3}\right) - 10.7 = 7.09931 \ldots$

The moment magnitude is approximately 7.1.