In the exercises that follow you should assume that $P$, $Q$, $R$ and $S$ are given statements that may be either true or false.

1. Write down the denial, the converse and the contrapositive form of each of the following statements:
   
   a. All cats scratch.
      
      The converse: All things that scratch are cats.
      The contrapositive: All things that do not scratch are not cats.
      The denial: There is at least one cat that does not scratch.
   
   b. If what you said yesterday is correct, then Jim has red hair.
      
      The converse: If Jim has red hair then what you said yesterday is correct.
      The contrapositive: If Jim does not have red hair then what you said yesterday is false.
      The denial: What you said yesterday is correct and Jim does not have red hair.
   
   c. If a triangle $\triangle ABC$ has a right angle at $C$ then
      
      $$(AB)^2 = (AC)^2 + (BC)^2.$$  
      
      The converse: Every triangle $\triangle ABC$ satisfying the the condition
      
      $$(AB)^2 = (AC)^2 + (BC)^2.$$  
      
      must have a right angle at $C$.
      
      The contrapositive: Every triangle $\triangle ABC$ that fails to satisfy the the condition
      
      $$(AB)^2 = (AC)^2 + (BC)^2.$$  
      
      will not have a right angle at $C$.
      
      The denial: There is at least one triangle $\triangle ABC$ that has a right angle at $C$ and for which the condition
      
      $$(AB)^2 = (AC)^2 + (BC)^2.$$  
      
      does not hold.
   
   d. If some cats scratch, then all dogs bite.
      
      The converse: If all dogs bite then some cats scratch.
      
      The contrapositive: If there exists a dog that does not bite then no cats scratch.
      
      The denial: Some cats scratch but not all dogs bite.
   
   e. It is with regret that I inform you that someone in this room is smoking.
      
      The converse: This statement doesn’t have a converse.
      
      The contrapositive: This statement doesn’t have a contrapositive form.
      
      The denial: It is without regret that I inform you that someone in this room is smoking.
   
   f. If a function is differentiable at a given number then it must be continuous at that number.
      
      The converse: If a function is continuous at a given number then it must be differentiable at that number.
      
      The contrapositive: If a function is not continuous at a given number then it cannot be differentiable at that number.
      
      The denial: There exists a function and a real number such that the function is differentiable at the number but fails to be continuous at that number.
g. Every boy or girl alive is either a little liberal or else a conservative.
   The converse: Every little liberal and every conservative is a living boy or girl
   The contrapositive: Any individual who fails to be either a little liberal or a conservative cannot be either a living boy or a living girl.
   The denial: There is at least one living boy or girl who is neither a little liberal nor a conservative.

2. In each of the following exercises, write down a denial of the given statement.
   a. All cats scratch and some dogs bite.
      The denial: Either there is a cat that does not scratch or no dogs bite.
   b. Either some cats scratch or if all dogs bite then some birds sing.
      The denial: There is at least one number x such that g(x) = 1.
   c. He walked into my office this morning, told me a pack of lies and punched me on the nose.
      The denial: Either he did not walk into my office this morning or he did not tell me a pack of lies or he did not punch me on the nose.
   d. No one has ever seen an Englishman who is not carrying an umbrella.
      The denial: At least one person has seen an Englishman who is not carrying an umbrella.
   e. For every number x there exists a number y such that x > y.
      The denial: There exists a number x such that for every number y we have y ≤ x.

3. In each of the following exercises we assume that f and g are given functions. Write down a denial of each of the following statements:
   a. Whenever x > 50, we have f(x) = g(x).
      The denial: There exists a number x > 50 such that f(x) ≠ g(x).
   b. There exists a number w such that f(x) = g(x) for all numbers x > w.
      The denial: For every number w there exists a number x > w such that f(x) ≠ g(x).
   c. For every number x there exists a number δ > 0 such that for every number t satisfying the condition |x - t| < δ, we have |f(x) - f(t)| ≥ 1.
      The denial: There exists a number x such that for every number δ > 0 there is at least one number t satisfying the condition |x - t| < δ such that |f(x) - f(t)| ≥ 1.
   d. There exists a number δ > 0 such that for every pair of numbers x and t satisfying the condition |x - t| < δ, we have |f(x) - f(t)| < 1.
      The denial: For every number δ > 0 it is possible to find a pair of numbers x and t satisfying the condition |x - t| < δ and for which |f(x) - f(t)| ≥ 1.
   e. For every ε > 0 and for every number x, there exists a number δ > 0 such that for every number t satisfying |x - t| < δ, we have |f(t) - f(x)| < ε.
      The denial: There exists a number ε > 0 and a number x such that for every number δ > 0 it is possible to find a number t such that |x - t| < δ and |f(t) - f(x)| ≥ ε.
   f. For every positive number ε there exists a positive number δ such that for every pair of numbers x and t satisfying the condition |x - t| < δ, we have |f(x) - f(t)| < ε.
      The denial: There exists a number ε > 0 such that for every number δ > 0 it is possible to find a pair of numbers x and t satisfying the condition |x - t| < δ for which |f(x) - f(t)| ≥ ε.
4. Explain why the statement \( \neg(P \Rightarrow Q) \) is equivalent to the statement \( P \land (\neg Q) \).
   
   The assertion \( P \Rightarrow Q \) says that if \( P \) is true then \( Q \) must also be true. This assertion says nothing at all about \( Q \) in the event that \( P \) is false. The only way in which the assertion \( P \Rightarrow Q \) can be false is that \( P \) is true and \( Q \) is not. In other words, the denial of the condition \( P \Rightarrow Q \) says that \( P \land (\neg Q) \).

5. Explain why the statement \( \neg(P \leftrightarrow Q) \) is equivalent to the assertion that either \( (P \) is true and \( Q \) is false) or \( (P \) is false and \( Q \) is true).
   
   The assertion \( P \leftrightarrow Q \) says that \( P \) and \( Q \) have the same truth value. So the assertion \( \neg(P \leftrightarrow Q) \) says that they don’t, which means that one of them is true and the other is false.

6. Explain why the statement \( \neg(P \lor Q) \) is equivalent to the statement \( (\neg P) \land (\neg Q) \).
   
   The assertion \( P \lor Q \) says that at least one of the statements \( P \) and \( Q \) is true. So the denial of the the assertion \( P \lor Q \) says that they are both false.

7. Explain why the statement \( \neg(P \Rightarrow (Q \lor R)) \) is equivalent to the assertion that both of the statements \( Q \) and \( R \) are false.
   
   The denial of the assertion \( P \Rightarrow (Q \lor R) \) says that \( P \) is true but that the assertion \( Q \lor R \) is false. In other words, it says that \( P \) is true and that both of the statements \( Q \) and \( R \) are false.

8. Explain why the converse of the statement \( P \Rightarrow (Q \lor R) \) is equivalent to the condition \( (R \Rightarrow P) \land (Q \Rightarrow P) \).
   
   The converse of the statement \( P \Rightarrow (Q \lor R) \) says that \( (Q \lor R) \Rightarrow P \) and this says that \( P \) must be true if at least one of the statements \( Q \) and \( R \) are true. In other words, the statement \( (Q \lor R) \Rightarrow P \) says that \( Q \Rightarrow P \) and also that \( R \Rightarrow P \).

9. Write the assertion \( P \Rightarrow (Q \lor R) \) as simply as you can in its contrapositive form.
   
   The contrapositive form of the assertion \( P \Rightarrow (Q \lor R) \) says that \( (Q \lor R) \Rightarrow \neg P \) and this says that if both of the statements \( Q \) and \( R \) are false then \( P \) is false. In other words, this contrapositive form says that \( ((\neg Q) \land (\neg R)) \Rightarrow (\neg P) \).

10. Write the assertion \( (P \land Q) \Rightarrow (R \lor S) \) as simply as you can in its contrapositive form.
    
    The contrapositive form of the assertion \( (P \land Q) \Rightarrow (R \lor S) \) says that
    
    \[ \neg(R \lor S) \Rightarrow \neg(P \land Q) \]
    
    which we can write as
    
    \[ (\neg R) \land (\neg S) \Rightarrow (\neg P) \lor (\neg Q) \].
    
    The latter statement says that if both of the statements \( R \) and \( S \) are false then at least one of the statements \( P \) and \( Q \) is false.