0.1 Fractals

0.1.1 What is a Fractal?

To start out this section on fractals we will begin by answering several questions. The first question one might ask is what is a fractal? In the 1970s, the mathematical Benoit Mandelbrot discovered a method to create geometric figures with special properties. This method allows figures to be enlarged repeatedly while preserving the detail of the figure. Mandelbrot refer to these figures as fractals. At the current time, we do not have universal agreement on an exact definition of a fractal. Based on the fractals we will study, We will define a fractal as a geometric figure that is divided into smaller versions of itself. A second common question that is asked is what does a fractal look like? This question is not as easy to answer because fractals can take on a wide range of patterns and designs, but the common element to all fractals is that they all contain repeating patterns. Below are some computer generated fractals that will give us an idea what a fractal looks like.

All of these picture where generated by Suzanne Alejandre they can view at this website: http://mathforum.org/alejandre/workshops/fractal/fractal3.html
A third question might be is how are fractals created? Usually fractals are made by starting with a general shape which is called an initiator. The initiator is then expanded out into different shapes by using what is called a generator. Here is an examples of how a fractal can be generated by using an initiator and a generator. This special fractal is known as the Koch curve.

**Example 1: The Koch Curve**

When producing the Koch Curve, we start with the given initiator and develop each step using the given generator. In this problem the generator is a line segment and the generator is created by dividing the segment into three equal segments and replacing the middle segment with a hump as shown in the below figure. The hump in the middle is formed by two segments that equal in length to the outer two segments.

![Initiator and Generator](image)

The first step or step 0 is the initiator which is a simple line segment. Next, step 1 you will apply the generator to the initiator which will result in a pattern identical to the initiator where there pattern consists of four line segments. (See below)

![Step 1](image)

On the next step (Step 2), you will apply the pattern of the initiator to each of the four line segment in step 1. The resulting pattern will look like the pattern below:

![Step 2](image)

Repeating this process for several steps will result in the fractal called the
To find the dimension of a fractal, we need to know three quantities. These quantities are known as the replacement ratio of the fractal, the scaling ratio of the fractal, and the magnification ratio of the fractal. The replacement ratio $N$ is the number of objects that replace the original object in the previous step. The scaling factor $s$ is the ratio between the new object and the original object. The scaling ratio $r$ is the reciprocal of the scaling ratio. The equation to find the dimension of the fractal is given $d = \frac{\log(N)}{\log(s)}$.

**Example 2: The dimension of the Koch Curve**

Recall that the Koch curve took one line segment and replaced it with four new line segments, so the replacement ratio would be $N = 4$. If you look at the original generator for the Koch curve, each new segment has a length that is one third the length of the original segment. Thus, the scaling factor would be $s = \frac{1}{3}$. The scaling ratio of the fractal would be the reciprocal of $s$, $r = \frac{1}{\frac{1}{3}} = 3$. Therefore, the dimension of the Koch curve would be $d = \frac{\log(4)}{\log(3)} = 1.26$.

**Example 3: The Koch Snowflake**

The Koch snowflake is similar to the Koch curve, except you use an equilateral triangle as an initiator. To generate the Koch Snowflake, you start with an equilateral triangle and use the Koch curve generator on each of the sides of the equilateral triangle.
In step 2, we repeat the pattern of the Koch curve on each side of the figure in step 1.

If we keep repeating this process will get a figure that resembles a snowflake.

Images courtesy thinkquest.org

**Example 4: Sierpinski’s Triangle**

To generate Sierpinski’s triangle, we start with a equilateral triangle and connect the midpoints of the three midpoints of the triangle. This will result in four congruent triangles as seen in step 1 below;

Equilateral triangle

Now, let’s shade in the outer three triangles and remove the middle triangle.
In step 2, you now take each of the three shaded triangles and repeat the process of divided the triangle into four equilateral triangles and removing the middle triangle.

Repeating this process will result in Sierpinski’s triangle

Example 5: The dimension of the Sierpinski’s Triangle
Recall that the Sierpinski’s triangle took one solid triangle and replaced it with three solid triangles of the next step, so the replacement ratio would be $N = 3$. If look at step 1 of Sierpinski’s triangle, each new side of new triangle has a length that is one half the length of the original triangle. Thus, the scaling factor would be $s = \frac{1}{2}$. The scaling ratio of the fractal would be the reciprocal of $r$. $r = \frac{1}{2} = 2$ Therefore, the dimension of the Koch curve would be $d = \frac{\log(3)}{\log(2)} = 1.46$
1. Find the dimension of the fractal given its scaling ratio $r$ and replacement ratio $N$. ($r = 4$ and $N = 5$)

2. Sketch the first two iterations of the fractal and then find its dimension.

   **Initiator**

   

   **Generator**

3. Find the dimension of the fractal given its scaling ratio $r$ and replacement ratio $N$. ($r = 5$ and $N = 7$).

4. Find the dimension of the following fractal given the initiator and generator of the fractal.

   **Initiator**

   

   **Generator**

5. Find the dimension of the fractal with the given initiator and generator.

   **The Hilbert Curve**

   **Initiator**

   

   **Generator**
6. Exploratory Question: Find the dimension of Serpinski’s Carpet. Compare the result to the dimension of the fractal in the above problem (The Hilbert Curve).