The program demoflow.f90 finds the numerical steady state solution for a hyperbolic system of equations. It is well-known that hyperbolic systems of PDEs pose special challenges. The fact that it is possible to get satisfactory solutions for many hyperbolic problems of physical interest attests to both the versatility of MOL and the quality of available ODE solvers. Yet, a key to a successful MOL solution is the spatial discretization. This is particularly true for hyperbolic equations. Since hyperbolic equations have natural characteristic directions associated with them, it is often necessary to use upwind spatial differences to handle equations with terms like \( v \frac{\partial u}{\partial z} \).

Such terms usually indicate movement of the solution with velocity \(|v|\) and with the direction of movement determined by the sign of \(v\); the spatial differencing should accommodate this movement.

Simple upwinding doesn’t always work however. It may be necessary to use different directional differences for the same term \( \frac{\partial u}{\partial x} \) in different equations of the system. We will illustrate this by using a “pseudo-characteristic” solution for the Navier-Stokes equations discussed earlier. The defining hyperbolic system of equations is given by

\[
\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial z} = C \quad (0 \leq t, 0 \leq z \leq L)
\]

where

\[
U = (\rho, G, T)^T
\]

\[
C = \left(0, -KG\frac{G}{\rho} - \rho g_a \sin(\theta), \frac{a^2 \Phi P_H \kappa}{C_p A_f}\right)^T
\]

and

\[
A = \begin{pmatrix}
0 & 1 & 0 \\
\frac{1}{\rho \kappa} - \frac{G^2}{\rho^2} & 2 \frac{G}{\rho} & \beta \\
-\frac{a^2 \beta T G}{\rho^2 C_p} & \frac{a^2 \beta T}{\rho C_p} & G
\end{pmatrix}
\]

Here \(G, \rho,\) and \(T\) are the flow rate, density, and temperature, respectively. \(L\) has the value 2.0953 corresponding to the saturation distance for this problem. We’ll use following boundary conditions:

\[
\rho(0, t) = \rho_0 = 795.521 \\
T(0, t) = T_0 = 255.000 \\
G(L, t) = G_0 = 270.900.
\]

What we wish to do is make an initial guess for the steady state solution, hold the boundary conditions at their constant values, and integrate in time until a numerical steady state solution is obtained. (This procedure is sometimes termed the method of “false transients.”) This is not an easy task. If the spatial discretization is not done carefully, a solver will not integrate to steady state. Even if a careful discretization is done many well-known ODE solvers have a really hard time with this problem. Before steady state is reached, \(G(x, t)\) is highly oscillatory. Once steady state is achieved, the resulting solution can be and typically is used as the initial solution for a new set of boundary conditions.

A straightforward calculation shows that the eigenvalues of \(A\) are

\[
\frac{G}{\rho}, \frac{G}{\rho} + a, \text{ and } \frac{G}{\rho} - a.
\]

The defining equations may be expressed in characteristic form by forming

\[
BAB^{-1} = D
\]
where $D$ is the diagonal matrix whose diagonal elements are these eigenvalues and $B$ is the matrix

$$rBA = \begin{pmatrix} \beta a^2 T & 0 & -\rho C_p \\ -Gka + 1 & \rho ka & \rho \beta \\ Gka + 1 & -\rho ka & \rho \beta \end{pmatrix}$$

The resulting system of equations is then

$$B \frac{\partial u}{\partial t} + DB \frac{\partial u}{\partial z} = BC.$$

At each spatial node $z_1, \ldots, z_{m+1}$ one sided differences are calculated for the spatial differences to be used in the corresponding equations of this system:

$$\rho_{z,0}, G_{z,0}, T_{z,0}$$
$$\rho_{z,+}, G_{z,+}, T_{z,+}$$
$$\rho_{z,-}, G_{z,-}, T_{z,-}$$

The subscripts 0, +, − indicate that the direction of the spatial differencing is dictated by the signs of the local characteristics $G\rho$, $G\rho + a$, and $G\rho - a$ at the spatial node. Backward differences are used if the sign of the characteristic is positive; otherwise forward differences are used.

At each spatial node, there results a linear system defined by

$$B \left( \frac{d\rho_i}{dt}, \frac{dG_i}{dt}, \frac{dT_i}{dt} \right)^T = E$$

where

$$E_1 = \sum_{j=1}^{3} B_{1j} C_j - \left( \frac{G_i}{\rho_i} \right) \{B_{11}\rho_{z,0} + B_{12}G_{z,0} + B_{13}T_{z,0}\},$$

$$E_2 = \sum_{j=1}^{3} B_{2j} C_j - \left( \frac{G_i}{\rho_i} + a \right) \{B_{21}\rho_{z,+} + B_{22}G_{z,+} + B_{23}T_{z,+}\},$$

$$E_3 = \sum_{j=1}^{3} B_{3j} C_j - \left( \frac{G_i}{\rho_i} - a \right) \{B_{31}\rho_{z,-} + B_{32}G_{z,-} + B_{33}T_{z,-}\}.$$ 

This system may be solved to yield the time derivatives for this spatial node.

A few comments regarding the characteristics of the resulting system of MOL ODEs are in order. The system is indeed stiff. One negative eigenvalue of the Jacobian matrix has magnitude approximately $O(10^5)$. The other eigenvalues are shifted into left half-plane (due to damping which is introduced implicitly into the system by the differencing scheme). The stiffness of the system roughly doubles each time the number of spatial nodes is doubled. Each of the solution components is oscillatory and damped with the oscillations in $G_i$ being as much as approximately ten times larger than the actual steady state solution values. In these respects the system exhibits characteristics of many time dependent problems.

It should be noted that if a nodewise ordering of the solution variables is used, the Jacobian for this system is banded. This is not done in the program demoflow.f90 in order to mimic the way in which sparse matrices arise in this context; for example, refer to demohum21.f90.

The program demoflow2.f90 uses a different set of boundary conditions:

$$\rho(L, t) = \rho_0 = 734.350$$
$$T(0, t) = T_0 = 255.000$$
$$G(0, t) = G_0 = 270.900.$$