Section 4.8 Combinations

Given a set of \( n \) objects, a combination of \( r \) of those objects is a subset of \( r \) objects, chosen without regard to order.

E.g., in how many different ways can a club with five members choose two members to go buy beer? (It doesn’t matter what order they’re chosen in, just which members are chosen.)

The number of ways of choosing a subset of \( r \) objects out of a set of \( n \) objects is denoted \( C(n, r) \) or \( \binom{n}{r} \). This is the number of combinations of \( r \) objects chosen from among \( n \) (or, sometimes, the number of combinations of \( n \) objects taken \( r \) at a time). Some books use \( _nC_r \) for \( C(n, r) \).

We can get at \( C(n, r) \) by breaking the task of counting the number of ways to choose \( r \) objects from among \( n \), in order, into two subtasks, then counting the number of ways to do each one.

First subtask: choose the \( r \) objects without regard to order. There are \( C(n, r) \) ways to do this (by definition).

Second subtask: order the \( r \) objects. There are clearly \( P(r, r) = r! \) ways to do this.

Thus there are \( C(n, r) \cdot r! \) ways to choose \( r \) objects from among \( n \) in a definite order. But we already know that there are \( P(n, r) \) ways to do this, so it must be true that \( C(n, r) \cdot r! = P(n, r) \), whence \( C(n, r) = \frac{P(n, r)}{r!} \). We already have a formula for \( P(n, r) \); when we plug it in and simplify we get

\[
C(n, r) = \frac{n!}{(n-r)!r!}
\]

Note that \( C(n, r) \) is the number of \( r \)-element subsets of an \( n \)-element set.

E.g., how many different 5-card hands can be dealt from a standard deck?
*E.g.*, a committee of ten wishes to select a subcommittee of four. How many such subcommittees are possible?

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*E.g.*, a string quartet is composed of a cellist, a violist, and two violinists. You have three cellists, two violists, and six violinists to choose from. (a) How many different string quartets can you make up?

(b) How many if the violinists are to be designated first and second violin?

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*E.g.*, how many full houses of three aces and two kings are there?
E.g., how many different 5-card hands consisting of two pair (and nothing better) can be dealt from a standard deck?

E.g., prove that $C(n, r) = C(n, n-r)$. 

Approaches to Counting Problems

The task of solving a counting problem is easier if you can identify its type. We’ve studied the following types.

Simple fundamental counting principle problems

In these you break a task into simple subtasks, count the number of ways to do each subtask, and multiply. Repetition may be allowed, as when choosing digits for a license plate or characters for a binary string.

Counting a union

In these you use the formula \( n(A \cup B) = n(A) + n(B) - n(A \cap B) \). These problems usually ask for the number of ways one thing or another could occur, *e.g.*, the number of ways to draw a king or a heart when drawing a single card.

Simple permutation problems

In these you must choose elements of a set in a definite order; you use \( P(n, r) = \frac{n!}{(n-r)!} \). Repetition cannot be allowed, as you are arranging distinct objects.

Permutations of indistinguishable objects

In these you must determine the number of arrangements of a collection of objects of different types, and you can’t tell objects of a type apart; you use the formula \( \frac{n!}{n_1!n_2!\cdots n_k!} \). These are generally easy to identify.

Simple combination problems

In these you must choose a subset of a given set of objects, and order doesn’t matter; you use \( C(n, r) = \frac{n!}{(n-r)!r!} \). Repetition cannot be allowed, because you are choosing a subset.

Complex problems

These are problems that must be broken into simpler problems before counting can be done. No general rule can be given; just be careful, and think hard.

The following three questions are often useful when thinking about simple counting problems:

1) How many do I have to choose from among?
2) How many am I choosing?
3) Does order matter?