Section 3.7 The Conditional

This section studies several statements closely related to the conditional \( p \rightarrow q \).

For an example, let \( p = \text{‘We play hard’} \) and \( q = \text{‘We win’} \) in the following table.

<table>
<thead>
<tr>
<th>Statement name</th>
<th>Symbols</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conditional</td>
<td>( p \rightarrow q )</td>
<td>If we play hard, we’ll win.</td>
</tr>
<tr>
<td>Converse (of ( p \rightarrow q ))</td>
<td>( q \rightarrow p )</td>
<td>If we win, we played hard.</td>
</tr>
<tr>
<td>Inverse (of ( p \rightarrow q ))</td>
<td>(~ p \rightarrow \sim q)</td>
<td>If we don’t play hard, we won’t win.</td>
</tr>
<tr>
<td>Contrapositive (of ( p \rightarrow q ))</td>
<td>(~ q \rightarrow \sim p)</td>
<td>If we didn’t win, we didn’t play hard.</td>
</tr>
</tbody>
</table>

Notice that the last three statements are defined relative to the conditional; for example, you cannot have a “contrapositive” all by itself—it must be the contrapositive of a particular conditional statement. Notice also that you sometimes have to change tenses in English to make conditional statements make sense.

E.g., state the converse, inverse, and contrapositive of (a) \( p \rightarrow \sim q \)

(b) ‘If Mary wins, I’ll be happy’
(c) ‘If the infected person is in good health, the illness is not as serious as normal pneumonia and there are rarely any complications’

E.g., show that \( \sim q \rightarrow \sim p \) is equivalent to \( p \rightarrow q \)

In English, conditionals are expressed in various ways. For example, we may hear that ‘It will rain only if the clouds move toward us’. If \( p = \)‘It will rain’ and \( q = \)‘The clouds move toward us’, does the speaker mean \( p \rightarrow q \), or \( q \rightarrow p \), or what? Here’s the analysis: ‘It will rain only if the clouds move toward us’, so, if the clouds don’t move toward us, it won’t rain. That is, if \( q \) doesn’t happen, then \( p \) won’t happen; \( i.e., \sim q \rightarrow \sim p \). Because \( \sim q \rightarrow \sim p \equiv p \rightarrow q \), we can see that ‘\( p \) only if \( q \)’ means \( p \rightarrow q \). Thus ‘It will rain only if the clouds move toward us’ says the same thing as ‘If it rains, then the clouds moved toward us’. Other expressions can be analyzed similarly.
Negating conditionals

First, $p \rightarrow q \equiv \sim p \lor q$ (truth table). We can use this to see how to negate $p \rightarrow q$:

$$\sim (p \rightarrow q) \equiv \sim (\sim p \lor q) \equiv p \land \sim q.$$  

The second equivalence is by DeMorgan’s Law. We get what we want by using the two ends of this chain of equivalences:

The negation of $p \rightarrow q$ is $p \land \sim q$.

This makes perfect sense; if $p =$ ‘We play hard’ and $q =$ ‘We win’ then $p \rightarrow q =$ ‘If we play hard, we’ll win’. Imagine someone asserting this to you: what would make him a liar? He’d be a liar if we play hard but we still don’t win, i.e., if $p \land \sim q$.

E.g., negate (a) $\sim p \rightarrow q$

(b) $p \rightarrow (\sim q \lor r)$

(c) $\sim (p \lor (q \rightarrow \sim r))$

E.g., negate: (a) ‘If roses are red, then violets are blue’

(b) ‘If roses are red and violets are blue, then today isn’t washing day’
(c) ‘John will go only if Mary goes’

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Necessary and sufficient

In the statement $P \rightarrow Q$, $P$ is said to be sufficient for $Q$, and $Q$ is said to be necessary for $P$.

E.g., (a) express the statement ‘Food is necessary for life’ as a conditional.

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(b) Write its negation in both words and symbols.

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E.g., (a) express the statement ‘George buys a new car whenever his odometer hits 100,000 miles’ as a conditional.

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(b) Write its negation in both words and symbols.
E.g., express the statement ‘Sufficient intake of Vitamin E is necessary, but not sufficient, for supple skin’ as a conditional.

The expression “$P$ is necessary and sufficient for $Q$” means $P \iff Q$. (This is also written “$P$ if and only if $Q$”.)