Differential Equations

Differential Equations are equations that contain an unknown function and one or more of its derivatives. Many mathematical models used to describe real-world problems rely on the use of differential equations (see examples on pp. 501-503).

Most of the differential equations we will study in this chapter involve the first order derivative and are of the form

\[ y' = F(x, y) \]

Our goal will be to find a function \( y = f(x) \) that satisfies this equation. The following two examples illustrate how this can be done for a basic differential equation and introduce some basic terminology used when describing differential equations.

**Example 1:** Find the general solution of the differential equation \( y' = 3x^2 \)

**Solution:**
The general solution (or family of solutions) has the form \( y = f(x) + C \), where \( C \) is an arbitrary constant. When a particular value concerning the solution (known as an initial condition) of the form \( y(x_0) = y_0 \) (read as \( y = y_0 \) when \( x = x_0 \)) is known, a particular solution, where a particular value of \( C \) is determined, can be found. The next example illustrates this.

**Example 2:** Find the particular solution of the differential equation \( y' = 3x^2, \ y(0) = 1 \).

**Solution:**

To check whether a given function is a solution of a differential equation, we find the necessary derivatives in the given equation and substitute. If the same quantity can be found on both sides of the equation, then the function is a solution.
**Example 3:** Determine if the following functions are solutions to the differential equation \( y'' - 2y' + 8y = 0 \).

a. \( y = e^x \)

b. \( y = 2e^{4x} \)

**Solution:**
Example 4: Verify that \( y = \sin x \cos x - \cos x \) is a solution of the initial value problem

\[
y' + (\tan x)y = \cos^2 x, \quad y(0) = -1, \quad \text{on the interval } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}.
\]

Solution:
Example 5: For what value of $r$ does the function $y = e^{rx}$ satisfy the differential equation $y' + y = 0$?

Solution: For the function $y = e^{rx}$ to be a solution, we must, after computing the necessary derivative, obtain the same quantities on both sides of the equation after substitution. For $y' + y = 0$, we must compute, using the chain rule applied to the exponential function of base $e$, $y' = re^{rx}$. Hence, we obtain

\[ y' + y = 0 \]
\[ re^{rx} + e^{rx} = 0 \] (Substitute for $y'$ and $y$)
\[ e^{rx} (r + 1) = 0 \] (Factor $e^{rx}$)
\[ \frac{e^{rx} (r + 1)}{e^{rx}} = 0 \] (Divide both sides by $e^{rx}$, which is allowable since $e^{rx} \neq 0$ for all $x$)
\[ r - 1 = 0 \] (Simplify)
\[ r = -1 \] (Solve)

Thus, $r = -1$ for $y = e^{rx}$ to be a solution.