Parametric Curves

We will investigate several aspects of parametric curves in the plane. The curve given by
\[ x(t) = \sin(t) \quad \text{and} \quad y(t) = \sin(2 \cdot t) \]
will serve as an example throughout, and we begin by defining the curve once and for all.
\[ x := t \rightarrow \sin(t); \quad y := t \rightarrow \sin(2 \cdot t); \]
\[ t \rightarrow \sin(t) \]
\[ t \rightarrow \sin(2 \cdot t) \]

Graphing

A simple modification of the `plot` command lets us graph a parametric curve: Enclose the two curves and the range of the parameter in a single bracket.

\[ plot( \{ x(t), y(t), t = 0..2 \cdot \text{Pi} \}, \text{scaling} = \text{constrained}) ; \]

The option `scaling=constrained` was added to obtain equal spacing on the coordinate axes.

Slope

**Problem:** Find the slope of the curve at the point where \( t = \pi/3 \).

**Solution:** Finding the derivative is easily accomplished using the formula \( \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \).
Thus the slope at \( t = \frac{\pi}{3} \) is -2.

### Graph of the Parametric Curve with a Tangent Line

**Problem:** Graph the curve along with its tangent line at the point where \( t = \pi/3 \) in a common figure.

**Solution:** The solution requires several steps. First we denote the point which corresponds to \( t = \pi/3 \) by \((X,Y)\), and we compute it as
\[
X := x\left(\frac{\pi}{3}\right) = \frac{1}{2} \sqrt{3} \\
Y := y\left(\frac{\pi}{3}\right) = \frac{1}{2} \sqrt{3}
\]

(1.3.1)

In order to find the slope, we substitute \( t = \pi/3 \) into the formula for \( m \), and call the result \( M \) (recall that \( m \) was determined above)
\[
M := \text{subs}\left(t = \frac{\pi}{3}, m\right); \quad \frac{2 \cos\left(\frac{2}{3} \pi\right)}{\cos\left(\frac{1}{3} \pi\right)}
\]

(1.3.2)

\[
M := \text{simplify}(M); \quad -2
\]

(1.3.3)

Thus the tangent line has slope \( m = -2 \).

We have several options to obtain the curve and the tangent line in a common figure. For one we could plot the curve and the tangent line separately and then superimpose the two with the display command. This approach requires uploading the plots package, and we will not apply this method here. As an alternative we can express the tangent line in parametric form and plot the two curves simultaneously, and we will take this route. A third option will be shown below.
The tangent line $y - Y = M(x - X)$ can be expressed in parametric form by setting $x = t$, and we get

$$y = Y + M(t - X)$$

This formula can be directly substituted into the plotting command. Result:

$$plot( \{ [x(t), y(t), t=0..\pi], [t, Y + M\cdot(t - X), t=0.5..1.2], scaling = constrained \};$$

Note that in the plotting command the two parametric curves were enclosed by brackets and that the command has the structure $plot( \{ [first curve], [second curve] \}, options);$

The third option is more elegant. We compute the tangent lines for the two functions $x(t)$ and $y(t)$ at $t=\pi/3$ as functions of $t$, denoted by $lx(t)$ and $ly(t)$, and then we plot $(lx(t),ly(t))$ as a parametric curve. Here are the details:

$$lx := t \mapsto X + D(x) \left( \frac{\pi}{3} \right) \cdot \left( t - \frac{\pi}{3} \right);$$

$$t \mapsto X + D(x) \left( \frac{1}{3} \pi \right) \left( t - \frac{1}{3} \pi \right) \tag{1.3.4}$$

$$ly := t \mapsto Y + D(y) \left( \frac{\pi}{3} \right) \cdot \left( t - \frac{\pi}{3} \right);$$

$$t \mapsto Y + D(y) \left( \frac{1}{3} \pi \right) \left( t - \frac{1}{3} \pi \right) \tag{1.3.5}$$

The next two pictures show that we indeed calculated ordinary tangent lines.

$$plot( [x(t), lx(t)], t=0..\pi);$$

$$plot([y(t), ly(t)], t=0..\pi);$$
Viewed as parametric curve, the pair $lx(t)$ and $ly(t)$ becomes the tangent line of our curve.

```matlab
plot([x(t), y(t), t = 0..2*Pi], [lx(t), ly(t), t = 0..2/Pi/3], scaling = constrained);
```

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**Find Points with a Given Slope**

**Problem:** Find the point in the first quadrant where the slope equals -1.

**Solution:** We set the slope $dy/dx = m$, and solve the equation $m = -1$ for $t$.

\[
m := \frac{D(y)(t)}{D(x)(t)};
\]

\[
\frac{2 \cos(2t)}{\cos(t)}
\]

\[
solve(m = -1, t);
\]

\[
\arccos\left(-\frac{1}{8} + \frac{1}{8} \sqrt{33}\right), \pi - \arccos\left(\frac{1}{8} + \frac{1}{8} \sqrt{33}\right)
\]

The first value appears to be the desired value for $t$, and we name it $T$ (copy-and-paste)

\[
T := \arccos\left(-\frac{1}{8} + \frac{1}{8} \sqrt{33}\right);
\]

\[
\arccos\left(-\frac{1}{8} + \frac{1}{8} \sqrt{33}\right)
\]
Then the point with slope $m=-1$ is located at $x(T)$ and $y(T)$. These points are:

\[ x(T) = \sqrt{1 - \left( -\frac{1}{8} + \frac{1}{8} \sqrt{33} \right)^2} \]  
\[ y(T) = \sin \left( 2 \arccos \left( -\frac{1}{8} + \frac{1}{8} \sqrt{33} \right) \right) \]  

Numerically the coordinates of the point with slope -1 are:

\[ \text{evalf}(x(T)); \quad \text{evalf}(y(T)); \]

\[
\begin{align*}
0.8660254040 \\
0.9550219348
\end{align*}
\]

Clearly, this point belongs to the first quadrant.

### Find the Slopes at the Origin

**Problem:** What are the slopes as the curve passes through the origin?

**Solution:** First we need to find the value of the parameter when the curve passes through the origin, that is, we must solve $x(t)=0$ and $y(t)=0$ simultaneously.

\[
\text{solve} \{ x(t) = 0, y(t) = 0 \}, t; \]

\[
\{ t = 0 \}, \{ t = \pi \}
\]

We find two values. Check:

\[
x(0), y(0); \quad x(\pi), y(\pi);
\]

\[
\begin{align*}
0, 0 \\
0, 0
\end{align*}
\]

For the slopes we find:

\[
\frac{D(y)(0)}{D(x)(0)}, \quad 2
\]

\[
\frac{D(y)(\pi)}{D(x)(\pi)}, \quad -2
\]

Thus the slopes are +2 and -2.

### Arc Length

**Problem:** What is the arc length of the curve?

**Solution:** We can implement the arc length formula directly, and we obtain using the templates on the left:

\[
\int_0^{2\pi} \sqrt{D(x)(t)^2 + D(y)(t)^2} \, dt
\]

\[
\int_0^{2\pi} \sqrt{\cos(t)^2 + 4 \cos(2t)^2} \, dt
\]  

(1.6.1)
Maple couldn't find an analytic solution, and we accept a numerical approximation in its place.

### Area

**Problem:** Find the area enclosed by the curve.

**Solution:** The area between a parametric curve and the x-axis can be computed as follows:

\[
A = \int_a^b y \, dx = \int_{t_1}^{t_2} y(t) \cdot x'(t) \, dt
\]

where \( t_1 \) and \( t_2 \) are to be selected to match the endpoints.

In our example we use symmetry, and multiply the area in the first quadrant by 4 to get the total area. The parameter interval is \([0, \pi/2]\), and the area becomes

\[
4 \cdot \frac{\pi}{2} \int_0^{\pi/2} y(t) \cdot D(x)(t) \, dt
\]

\[
\frac{8}{3}
\]

\[\text{(1.7.1)}\]

### Polar Coordinates

*restart;*

Here we use the example \( r(\theta) = 3 + 5 \sin \theta \), and we shall work several problems related to this curve. First, we define \( r \) once and for all

\[
r := t \rightarrow 3 + 5 \cdot \sin(t);
\]

\[\text{(2.1)}\]

It is easier to work with \( t \) rather than typing \( \theta \) all the time. However, since \( r \) is a function (change the input from \( t \) to \( \theta \)) we have

\[
r(\theta);
\]

\[\text{(2.2)}\]

### Polar Graphs

Plotting in polar coordinates requires the option `coords=polar`, everything else is pretty much the same. This is the equivalent of setting your calculator to the polar function mode. Here is the graph:

\[
\text{plot}( r, 0 .. 2\cdot\Pi, \text{coords} = \text{polar}, \text{scaling} = \text{constrained});
\]
The plots package offers a nice alternative. In order to use it, we need to activate it first with the command

\texttt{with(plots)}:

Use a colon to avoid unnecessary output. Now we are ready to plot.

\texttt{polarplot(r(t), t = 0..2*\pi)};

\underline{Slope at a Given Point}

\textbf{Problem:} What is the slope of the curve at the point \((x,y)=(3,0)\)?

\textbf{Solution:} First of all we realize that the point \((3,0)\) is obtained by setting \(\theta = 0\). For the slope \(dy/dx\) we follow the usual procedure for parametric curves, this time with the \texttt{diff} command

\[ \frac{\text{diff}(r(t) \cdot \sin(t), t)}{\text{diff}(r(t) \cdot \cos(t), t)} \]

\[ \frac{5 \cos(t) \sin(t) + (3 + 5 \sin(t)) \cos(t)}{5 \cos(t)^2 - (3 + 5 \sin(t)) \sin(t)} \]

\[ (2.2.1) \]
And now we set \( t = 0 \)

\[
\frac{\cos(0) (10 \sin(0) + 3)}{10 \cos(0)^2 - 3 \sin(0) - 5}
\]  

(2.2.3)

\[
\frac{3}{5}
\]  

(2.2.4)

Thus the desired slope is 3/5.

### Graph of the Polar Curve with a Tangent Line

**Problem:** Graph the curve along with its tangent line at the point (3,0) in a common figure.

**Solution:** We know from the last problem that the slope is 3/5, and using the point-slope form of a line, the tangent line is given by

\[
y = \frac{3}{5} (x - 3)
\]

For the graph we have two alternatives: Either express the line in polar coordinates and use a single plotting statement, or graph the two pieces separately and superimpose the graphs. We will demonstrate both.

In polar coordinates the line becomes

\[
r \cdot \sin(t) = \frac{3}{5} \cdot (r \cdot \cos(t) - 3);
\]

and solving this equation for \( r \) we find

\[
solve(\%, r);
\]

\[
\frac{9}{-5 \sin(t) + 3 \cos(t)}
\]  

(2.3.2)

This is an expression for the tangent line in polar coordinates, and with a simple copy-and-paste we obtain

\[
plot \left( \left[ r(t), \frac{9}{-5 \sin(t) + 3 \cos(t)} \right], t = -\text{Pi}..\text{Pi}, \text{coords} = \text{polar}, \text{view} = [-5..6,-2..9], \text{scaling} = \text{constrained} \right);
\]
The alternative requires the plotting package (which is already activated).

\[ A := \text{polarplot}(r, 0..2\cdot\text{Pi}) : \# \text{store the graph of the curve as } A \]
\[ B := \text{plot}\left(\frac{3}{5} \cdot (x - 3), x = 0..6\right) : \# \text{save the tangent line in rectangular coordinates as } B \]
\[ \text{display}(A, B) ; \# \text{superimpose} \]

\begin{align*}
\text{Slopes at the Origin} \\
\text{Problem:} & \quad \text{What are the slopes of the curve as it passes though the origin?} \\
\text{Solution:} & \quad \text{First we set } r=0 \text{ to find the points where the curve passes through the origin.} \\
& \quad \text{solve}(r(t) = 0, t) ; \\
& \quad \text{arcsin}\left(\frac{3}{5}\right) \quad \text{(2.4.1)}
\end{align*}

This result is of limited use. But applying our knowledge of trigonometry we conclude that \( r(t)=0 \).
for

\[ t = 2\pi - \arcsin(3/5) \quad \text{and} \quad t = \pi + \arcsin(3/5) \]

Let's confirm this:

\[ T1 := \pi + \arcsin\left(\frac{3}{5}\right); \quad r(T1); \]

\[ \pi + \arcsin\left(\frac{3}{5}\right) \]

0

\[ (2.4.2) \]

\[ T2 := 2\cdot\pi - \arcsin\left(\frac{3}{5}\right); \quad r(T2); \]

\[ 2\pi - \arcsin\left(\frac{3}{5}\right) \]

0

Now we calculate the slope at these points.

\[
\frac{\frac{d}{dt}(r(t) \cdot \sin(t))}{\frac{d}{dt}(r(t) \cdot \cos(t))} = \frac{5 \cos(t) \sin(t) + (3 + 5 \sin(t)) \cos(t)}{5 \cos(t)^2 - (3 + 5 \sin(t)) \sin(t)}
\]

\[ (2.4.4) \]

\[ \text{subs}(t = T1, \%); \]

\[
\left(5 \cos\left(\pi + \arcsin\left(\frac{3}{5}\right)\right)\right) \sin\left(\pi + \arcsin\left(\frac{3}{5}\right)\right) + \left(3 + 5 \sin\left(\pi + \arcsin\left(\frac{3}{5}\right)\right)\right) \cos\left(\pi + \arcsin\left(\frac{3}{5}\right)\right) \left(5 \cos\left(\pi + \arcsin\left(\frac{3}{5}\right)\right)^2 - (3 + 5 \sin(t)) \sin(t)\right)^2 - (3 + 5 \sin(t)) \sin(t)
\]

\[ (2.4.5) \]

\[ \text{simplify}(\%); \]

\[ \frac{3}{4} \]

\[ (2.4.6) \]

\[
\frac{\frac{d}{dt}(r(t) \cdot \sin(t))}{\frac{d}{dt}(r(t) \cdot \cos(t))} = \frac{5 \cos(t) \sin(t) + (3 + 5 \sin(t)) \cos(t)}{5 \cos(t)^2 - (3 + 5 \sin(t)) \sin(t)}
\]

\[ (2.4.7) \]

\[ \text{simplify}(\text{subs}(t = T2, \%)); \]

\[ -\frac{3}{4} \]

\[ (2.4.8) \]

There is a Much easier way to determine the slope. We know from class that the slope is the tangent of the respective angles.

\[ \tan(T1); \tan(T2); \]

\[ \frac{3}{4} \]

\[ -\frac{3}{4} \]

\[ (2.4.9) \]
\textbf{Arc Length}

\textbf{Problem:} Find the arc length of the curve.

\textbf{Solution:} Here we implement the familiar formula for the arc length of a polar curve directly, and let maple do the work.

\[ \int_0^{2\pi} \sqrt{r(t)^2 + D(r(t))^2} \, dt \]

\[ 48 \text{EllipticE}\left(\frac{1}{4} \sqrt{15}\right) - \text{EllipticPi}\left(\frac{4}{17} \sqrt{17}, \frac{15}{16}, \frac{1}{4} \sqrt{15}\right) - 16 \text{EllipticE}\left(\frac{1}{2} \sqrt{2}\right), \quad (2.5.1) \]

\[ \frac{1}{4} \sqrt{15} \]

\[ \text{at 10 digits} \quad 34.31368713 \quad (2.5.2) \]

The result involves special functions, but numerical evaluation leads to an acceptable result.

\textbf{Area}

\textbf{Problem:} Compute the area enclosed by the inner loop of the curve.

\textbf{Solution:} The inner part of the curve is traced when \( T_1 < t < T_2 \) (the numbers \( T_1 \) and \( T_2 \) were found above). Using the area formula for polar curves we find that

\[ \frac{1}{2} \cdot \int_{T_1}^{T_2} r(t)^2 \, dt \]

\[ \frac{43}{4} \pi - \frac{43}{2} \arcsin\left(\frac{3}{5}\right) - 18 \quad (2.6.1) \]

\texttt{evalf(%)}; \quad 1.93684719 \quad (2.6.2)

The area is roughly two square units.