Shrinking circle, spheres and etc.

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In the Calculus textbook by J. Stewart, there is one interesting problem about limit: If we look at the figure $C_1 : (x-1)^2 + y^2 = 1$ and a shrinking circle C_2 with radius r and center the origin. Let P be the point $(0, r)$, and $Q = (x_2, y_2)$ be the upper point of intersection of the two curves mentioned above. If we let R be the point of intersection of the line PQ and the $x-axis$. What happens to R as C_2 shrinks to $(0,0)$, or as $r \to 0^+$.

Explorations

1. Click **here** for ClassPad animation.
2. Click **here** for the curve C_1 is replaced by an ellipse; Thanks Nicholas Jackiw and Daniel Scher.
3. Some numerical glitches.
4. **Paper** by Rober Kreczner.
5. Click here for a **shrinking sphere** (Cabri 3D).
6. Click **here** for a 3D scenario by Douglas Mead.
7. Clcik **here** for a 3D scenario by Kevin Thompson.

Key formula

Assume now C_1 satifies $F(x,y) = 0$. The line equation which connects the top of C_2 and the top intersection of between C_1 and C_2 should be

$$y - r = m(x-0), \qquad \#$$

where $m = \frac{y_2 - r}{x_2}$.

The $x-intercept$ of the line

$$x = -\frac{r}{m} = -\frac{rx_2}{y_2 - r}. \qquad \#$$

1. When C_1 is a circle, it is not hard to find the intersection in rectangular coordinate form which gives us (link to Maple)

$$\left\{x = \frac{r^2}{2}, y = \frac{\sqrt{4-r^2}}{2} r\right\} \qquad \#$$

2. We get the slope to be

$$m = \frac{\frac{\sqrt{4-r^2}}{2} r - r}{\frac{r^2}{2}} = \frac{1}{r}\left(\sqrt{4-r^2} - 2\right) \qquad \#$$

3. We get the line equation to be $(y - r) = m * (x - 0)$.

$$y = r + mx = \frac{1}{r}\left(x\sqrt{4-r^2} - 2x + r^2\right) = \frac{1}{r}\left(x\sqrt{4-r^2} - 2x + r^2\right) \qquad \#$$

4. We find the $x-intercept$ of line to be $\dfrac{-r^2}{\sqrt{4-r^2}-2}$.
5. Finally, we let $r \to 0^+$ and compute symbolically, we get

$$\lim_{r\to 0^+} \frac{-r^2}{\sqrt{4-r^2}-2} = 4. \qquad \#$$

Finding intersection of two curves is not easy

Use polar coordinate instead of rectangular coordinate

1. We use $x = r\cos\theta$ and $y = r\sin\theta$ for the curve C_2 and follow the steps above to find the x intercept and let $r \to 0^+$ and get the answer to be 4. See Method 2 here using Maple.
2. Example: When C_1 is not symmetric to the $x-axis$. Click here and see example 2.
3. When C_1 is a cubic function: Click here and go to example 3.
4. When C_1 is a horizontal parabola: Click Example 4.

Example *When C_1 is a hyperbola. Click Example 5. Consider $a = 2$ and $b = 2$ for the following Hyperbola:*

$$\frac{(x+a)^2}{a^2} - \frac{y^2}{b^2} = 1. \qquad \#$$

5. What about the shrinking circle is replaced by a shrinking ellipse.
   - **a**. Scenario 1.
   - **b**. Scenario 2.