A mathematics competition problem of Hubei high school of China in 1992.

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Example *Find the maximum of* $f(x) = \sqrt{x^4 - 3x^2 - 6x + 13} - \sqrt{x^4 - x^2 + 1}$.

Solution:

Rewrite $f(x)$ as $f(x) = \sqrt{(x-3)^2 + (x^2-2)^2} - \sqrt{(x-0)^2 + (x^2-1)^2}$. And let $A = (3,2), B = (0,1)$, then $f(x) = PA - PB$, where P is a point on the parabola $y = x^2$.

1. If $f(x) = \left|\sqrt{(x-a)^2 + (x^2-b)^2} - \sqrt{(x-c)^2 + (x^2-d)^2}\right|$ and $A = (a,b)$ and $B = (c,d)$, then find the maximum of $f(x)$ $[= PA - PB.]$.

   Remark *This is a special case of item 2 below.*

2. If $f(x) = \left|\sqrt{(x-a)^2 + (g(x)-b)^2} - \sqrt{(x-c)^2 + (g(x)-d)^2}\right|$ and $A = (a,b)$ and $B = (c,d)$, then find the maximum of $f(x)$ $[= PA - PB.]$

Three Cases:

Case 1. When AB intersects $y = g(x)$

Refer to the **ClassPad file**, (journal file)

Key: Since $PA - PB < AB$ (refer to the journal file), **the maximum value of** $f(x)$ **happens when the slopes of** PA, PB **and** AB **are identical**. (In the CP, A is the point on the curve, and there are two points that are not on the curve).

- $g(x) = (x+1)(x-1)(x-3)$
- $\frac{g(x)-2}{x-1} = \frac{g(x)-3}{x+1}$, Solution is: $\{[x = 3.1091]\}$

Case 2. When segment AB does not intersect with $y = g(x)$ but the line AB does.

Refer to the ClassPad file. The maximum value of $y = f(x)$ is different from finding the minimum area of the triangle PAB.

Case 3. When the line AB does not intersect with $y = g(x)$.

Refer to the ClassPad file case 3.

A different problem

Two dimensional case

Given (arbitrary) two differentiable functions f and g in some interval, we want to show that for all points (**in some neighborhood** of $y = f(x)$), there is a corresponding point on $y = g(x)$ so that the distance is the shortest between these two curves.

Key: If we start with one point from $y = f(x)$, the slope connecting the desired two points should be negative reciprocal to the slope of the tangent line at $x = c$.

1. We find $g'(x)$.
2. We solve for c so that the following equation is met

$$\left(\frac{g(c) - f(x)}{c - x}\right) \cdot g'(c) = -1.$$

- Sometimes, we can solve for c explicitly in terms of x (**The inverse function theorem**) When c can be solved explicitly in terms of x, say $c = G(x)$, then $y = G(x)$ is the solution curve. For example, given x_0 on the curve $y = f(x)$, we find the corresponding point on $y = g(x)$ by using $(G(x_0), g(x_0))$.
- If c can not be explicitly solved, then the solution c that we are looking for satisfying the following implicit equation

$$\left(\frac{g(c) - f(x)}{c - x}\right) \cdot g'(c) + 1 = 0.$$

('**Implicit function theorem**')
(http://www.ualberta.ca/dept/math/gauss/fcm/calculus/multvrbl/basic/ImplctFnctns

3. If x is given, we should be able to find c from the above equation.

Example **Let C_1 be the circle represented by $x^2 + (y-1)^2 = 1$ and C_2: $g(x) = -x^2 - 1$. Use **ClassPad** to do animation (open distance06-2)*

Example *(Explicit solution) Let $f(x) = \sqrt{1-x^2}$ and $g(x) = -x^2 - 1$. Link to **Maple**. Use **ClassPad** (open distance06-5) to do animation.*

Example *Let C_1 : $f(x) = x^2 + 1$ and C_2 : $g(x) = -x^2 + 1$. Use **ClassPad** (open **distance06-6**) to do animation.(2d Link to **Maple**)*

Example *(Implicit solution) Let $f(x) = x + \sin(x)$ and $g(x) = \tan x$. (link to **Maple**).Use **ClassPad** (**open example4**) to do animation: The equation of*

$$\left(\frac{g(c) - f(x)}{c - x}\right) \cdot g'(c) + 1 = 0. \qquad \#$$

becomes

$$\frac{(-\tan(c) + x + \sin(x))(1 + (\tan(c))^2)}{-c + x} + 1 = 0. \qquad \#$$

Example **(Implicit solution) Let C_1 : $f(x) = \sin(x) + 2$ and C_2 : $g(x) = -\cos x - 2$. (2d Link to **Maple**) (A **journal** file)*

Example *(Implicit solution) Let C_1 : $f(x) = x + \sin(x)$ and C_2 : $g(x) = -\cos x - 1$.(Link to **Maple**)*

Three dimensional case

Given the following (arbitrary) two **continuously differentiable** functions: $z = f(x,y)$ and $z = g(x,y)$, we want to show that for all points A on some neighborhood of $z = f(x,y)$, we can find a corresponding point B on $z = g(x,y)$ so that the distance AB is the shortest between these two surfaces. Note that if we write $G(x,y,z) = z - g(x,y) = 0$. The level curve of G at level 0 is precisely the curve $z = g(x,y)$. The graident of G is

$$\nabla G = \left(\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial z}\right) = (-g_x, -g_y, 1). \qquad \#$$

Key for solving this problem: Let $A = (x,y,f(x,y))$ and $B = (c,d,g(c,d))$. Then

$$AB = k \cdot (\nabla G)|_{(c,d)}. \qquad \#$$

1. Equation 1: $c - x = k(-g_x)|_{(c,d)}$
2. Equation 2: $d - y = k(-g_y)|_{(c,d)}$
3. Use Equation 1 and Equation 2 to solve for c and d in terms of x,y and k.
4. Equation 3: $g(c,d) - f(x,y) = k$.
5. Plug c and d into Equation 3 and get an implicit equation of x,y and k. We plot this 3d implicit equation.
6. By looking at the 3d implicit plot, we will see what (x,y) are suitable for finding (c,d).
7. By pluging suitable x and y, we find k.
8. **The exisitence of k is guaranteed by **Implicit Function Theorem**??
9. And we use Equation 1 and Equation 2 to find c and d correspondingly.

Example *If* $f(x,y) = xy + 5$ *and* $g(x,y) = x^2 + y^3 - 5$. *Click* ***here***.

Example *If* $f(x,y) = xy$ *and* $g(x,y) = x^2 + 2$. *Click* ***here***.

Example *If* $f(x,y) = -x^2 - y^2 - 2$ *and* $g(x,y) = (x-1)^2 + (y-1)^2$. *Click* ***here***.

Example *If* $f(x,y) = \sin x \cos y - 2$ *and* $g(x,y) = x^2 + y^2$. *Click* ***here***.

Orthogonal mapping

Suppose now we pick a space curve on $z = f(x,y)$, say $S_1 = [s, s^2, f(s,s^2)]$, we want to find the corresponding curve S_2 on $z = g(x,y)$ so that the vector connecting each point A on S_1 and the corresponding point B on S_2 is parallel to the normal vector of the tangent plane at B.

Example **If* $f(x,y) = xy$, $S_1 = [s, s^2, f(s,s^2)]$ *and* $g(x,y) = x^2 + 2$. *Click* ***here***.

Example **If* $f(x,y) = -x^2 - y^2 - 2, S_1 = [s, s^3, f(s,s^3)]$, *and* $g(x,y) = (x-1)^2 + (y-1)^2$. *Click* ***here***.

Example *If* $f(x,y) = xy + 5, S_1 = [s, s^2, f(s,s^2)]$ *and* $g(x,y) = x^2 + y^3 - 5$. *Clcik* ***here***.