Cauchy Mean Value Theorem, its converse and Largrange Remainder Theorem

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The Cauchy Mean Value Theorem.

Suppose the function $f : [a,b] \to R$ and $g : [a,b] \to R$ are continuous and that their restrictions to (a,b) are differentiable. Moreover, assume that $g'(x) \neq 0$ for all x in (a,b). Then there is a point x_0 in (a,b) at which

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x_0)}{g'(x_0).}$$

Remark : This is **not intuitive** at all, we should see how this can be made eaiser to understand from geometric point of view.

Parametric Form

Consider a curve of the form $r = h(\theta)$ or the form $[x(\theta), y(\theta)] = [r\cos\theta, r\sin\theta] = [g(\theta), f(\theta)]$. If the curve is a smooth curve in the interval $[a,b]$, it follows from MVT that we can find $\theta \in (a,b)$ so that

$$\frac{dy}{dx}|_\theta = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)}{g'(\theta)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

This implies that $f'(\theta)(g(b)-g(a)) = g'(\theta)(f(b)-f(a))$;integrating over θ on both sides, we get

$$f(\theta)(g(b)-g(a)) - g(\theta)(f(b)-f(a)) = \text{constant.}$$

If we define $F(x) = f(x)(g(b)-g(a)) - g(x)(f(b)-f(a))$, we will see $F(a) = F(b)$, and we may apply the Rolle's theorem on F. This gives us a glimpse how we prove the Cauchy Mean Value Theorem.

Converse of Mean Value Theorem

Theorem *(Known) Suppose f' is strictly monotone in the interval $[a,b]$. Then there exists $x_0 \in (a,b)$ such that*

1. *When $x \in (a,x_0)$, there exists a unique $p \in (a,b)$ satisfying*

$$f'(x) = \frac{f(p) - f(a)}{p - a}$$

2. *When* $x \in (x_0, b)$, *there exists a unique* $p \in (a,b)$ *satisfying*

$$f'(x) = \frac{f(b) - f(p)}{b - p}.$$

Remark:

- *Notice that in the previous theorem,* f *is either concave up or concave down in some interval. Also, we have to find the point* x_0 *where we apply the converse of MVT. To demonstrate this, I use CP,* ***click here****.*
- *I propose the following which is an extension of the known theorem:*

Theorem *(Yang) Suppose f is differentiable on* (a,b) *and continuous on* $[a,b]$. *Then*

1. *When the left end point* $x = a$ *is fixed, we can find* $x_0 \in (a,b)$ *such that for all* $x \in (a, x_0)$, ***we can find*** $p \in (a,b)$ *such that*

$$f'(x) = \frac{f(p) - f(a)}{p - a}. \qquad \#$$

2. *When the right end point* $x = b$ *is fixed, we can find* $x_0 \in (a,b)$ *such that for all* $x \in (x_0, b)$, ***we can find*** $p \in (a,b)$ *such that*

$$f'(x) = \frac{f(b) - f(p)}{b - p} \qquad \#$$

Converse of Cauchy Mean Value Theorem

Theorem *(Stanciu. T.N. Bul. Univ. Galati Fasc. II Mat. Fiz. Mec. Teoret. 3 (1980), 47-48. Let* f, g*:* $I \to R$ *be differentiable functions; I is an interval. Let* $g'(i) > 0$ *or* $g'(i) < 0$ *for all* $i \in I$*.Let further* $\psi : I \to R$, $\psi(z) = \frac{f'(z)}{g'(z)}$ *and* $M = \{(x,y) \in R \times R : x \in I, y \in I, x < y\}$. *If* $\psi(c)$ *belongs within the bounds for* $\psi(z), z \in I,$ *then* ***there exists*** $(a,b) \in M$ *with*

$$\frac{f(a) - f(b)}{g(a) - g(b)} = \psi(c).$$

Conjecture *Let* f, g*:* $I \to R$ *be differentiable functions on an interval* (a,b). *Let* $g'(x) > 0$ *or* $g'(x) < 0$ *for all* $x \in (a,b)$*.We* ***can find*** *the largest subintervals* (a,c) *or* (d,b) *of* (a,b) *so that*

1. *If* $x \in (a,c)$, ***we can find*** $x_0 \in (a,b)$ *such that*

$$\frac{f'(x)}{g'(x)} = \frac{f(x_0) - f(a)}{g(x_0) - g(a)}.$$

2. *If* $x \in (d,b)$, ***we can find*** $x_0 \in (a,b)$ *such that*

$$\frac{f'(x)}{g'(x)} = \frac{f(b) - f(x_0)}{g(b) - g(x_0)}.$$

Conjecture *Suppose the function* $f : [a,b] \to R$ *and* $g : [a,b] \to R$ *are continuous and that their restrictions to* (a,b) *are differentiable. Moreover, assume that* $g'(x) \neq 0$ *for* *all x in (a,b). **Then there exists** $x_0 \in (a,b)$ such that*

1. *When $x \in (a,x_0)$, there exists a unique $p \in (a,b)$ satisfying*

$$\frac{f'(x)}{g'(x)} = \frac{f(p)-f(a)}{g(p)-g(a)}$$

2. *When $x \in (x_0,b)$, there exists a unique $p \in (a,b)$ satisfying*

$$\frac{f'(x)}{g'(x)} = \frac{f(b)-f(p)}{g(b)-g(p)}.$$

Remark *To understand the CMV (formed by f and g), we write the curve by using $[x(t),y(t)] = [g(t),f(t)]$. This will help us understand (only) what the curve would look like*

Remark *But to see if the converse of CMV holds, we need to use the Conjecture above, we treat f and g as separate functions. We apply two new functions*

$$F(x) = \frac{f'(x)}{g'(x)} \qquad \#$$

and

$$G(x) = \frac{f(x)-f(a)}{g(x)-g(a)} \text{ or } G(x) = \frac{f(b)-f(x)}{g(b)-g(x)} \qquad \#$$

and find the interval where the converse of CMV holds.

1. *Sketch $y = F(x)$ and $y = G(x)$.*
2. ***Analyze** the graphs and find the largest subinterval where converse of CMV holds.*
   a. *Roughly speaking, find the intersection of $y = F(x)$ and $y = G(x)$, say (xc,yc).*
   b. *Set $F(x) = yc$ and solve for x, say x_0.*
   c. *The converse of CMV holds for (x_0,b) or (a,x_0).*

Examples

Example **Let $f(x) = -x^2+1$ and $g(x) = x^3+x$ over the interval $[-4,4]$. Click **here** for*

*ClassPad and open '**parametric-CMV**'-Example 1. For Maple, click **here**.*

Example **Let $f(x) = x^2-x+x^3$, and $g(x) = 2x-\cos x$ and assume $[a,b] = [-4,4]$. For Maple, click **here**. For CP, click **here** and open 'parametric-CMV'-Example 2.*

Example *For $r = \theta$, the polar equation can be written as*

$$[f(t),g(t) \qquad \#$$

or

$$[t\sin(t),t\cos(t)]. \qquad \#$$

*Click **here** for CP. Click **here** for Maple.*

Example *Let $f(x) = x+2x^2\sin(1/x)$ and $g(x) = \cos(x)-2x$, and we consider the functions in the interval $[-1,1]$. Notice that $f'(x)$ is oscillating around $x = 0$ and*

*$g'(x) < 0$. Click here for **Maple**. Click here for CP.*

- *When the right point $x = 1$ is fixed, the intersection of $F(x) = \frac{f'(x)}{g'(x)}$ and $G(x) = \frac{f(1)-f(x)}{g(1)-g(x)}$ is at $x = 0.3264227762$. But it follows from the Maple worksheet above that the converse of CMV (when right end point is fixed) does not hold for the function F(x) in the interval x in (0.3264227762,1). But we can find a subinterval of (0.3264227762,1) so that the converse of CMV holds for F there. It follows from the Maple worksheet that the converse of CMV holds for F in the interval (0.6797271506,1).*
- *When the left end point $x = -1$ is fixed, we have the following observations:*
- *We don't know if the converse of CMV for F holds for the interval (-1,0.3296655172). But we can prove that there is a subinterval of (-1,0.3296655172) so that the converse of CMV for F holds there. Refer to Maple worksheet, This is what we need. Therefore the converse of CMV holds for F on the interval $(-1, -.3005927755)$.*

Exercise *Let $f(x) = e^x$ and $g(x) = -\cos(x) + 2x$ on $(-1, 1)$. [Note that f and g are strictly increasing in $(-1, 1)$, and $g'(x) = 2 + \sin(x) > 0$.]*

Exercise *Let $f(x) = \tan(x)$ and $g(x) = -\cos(x) + 2x$ on $(-1, 1)$. [Note that f is strictly increasing in $(-1, 1)$ and $g'(x) = 2 + \sin(x) > 0$.]*

Exercise *Let $f(x) = \tan(x)$ and $g(x) = \cos(x) - 2x$ on $(-1, 1)$. [Note that f is strictly increasing in $(-1, 1)$ and $g'(x) = -2 - \sin(x) < 0$.]*

Real life interpretations

1. Cauchy Mean Value Theorem, Inequality Version. If functions f and g are continuous on $[a, b]$ and differentiable on (a, b), and if g is strictly monotone, then

$$m \leq \frac{f'(x)}{g'(x)} \leq M,$$

implies

$$m \leq \frac{f(b) - f(a)}{g(b) - g(a)} \leq M.$$

2. Mean Value Theorem, Inequality Version.

$$m \leq f'(x) \leq M,$$

implies

$$m \leq \frac{f(b) - f(a)}{b - a} \leq M.$$

- If a car A is going at a speed between 50 and 60 units (say miles per hour) in 3 hours ($b - a = 3$) then the car has traveled between 150 and 180 miles in 3 hours.

3. A Physical Application: If car A is going at a speed which is between 2 and 3 times the speed of car B, then car A will have traveled between 2 and 3 times of the distance of car B. [R. Vencil Skarda of Brigham Young University]

The Largrange Remainder Theorem

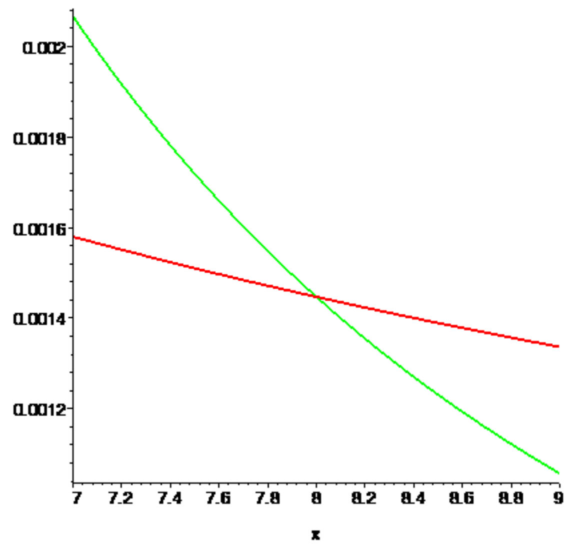
Theorem *Let I be a neighborhood of the point x_0 and let n be a nonnegative integer. Suppose that the function $f : I \to R$ has $n+1$ derivatives. Then for each $x \neq x_0$ in I, there is a point c in (x, x_0) or (x_0, x) such that*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1.}$$

Example *Consider the function $f(x) = x^{\frac{1}{3}}$, and we pick $x_0 = 8$, verify the Largrange Remainder Theorem for the function f at x_0, $I = (7,9)$ and $n = 2$.*

Solution *Link to **Maple**.*

1. *The 2nd degree Taylor polynomial for f at $x_0 = 8$ is*
   $p2(x,8) = p_2(x) = \sqrt[3]{8} + \frac{(x-8)\sqrt[3]{8}}{24} - \frac{(x-8)^2\sqrt[3]{8}}{576}$.
2. *The Remainder*
   $R_2(x,8) = R_2(x) = f(x) - p_2(x) = \sqrt[3]{x} - \left(\sqrt[3]{8} + \frac{(x-8)\sqrt[3]{8}}{24} - \frac{(x-8)^2\sqrt[3]{8}}{576}\right).$
3. *We want to show that for for each $x \neq 8$ in $(7,9)$, there is a point c in $(x,8)$ or $(8,x)$ there is a c in (x,x_0) such that*
   $$R_2(x) = \frac{R_2^{(3)}(c)}{3!}(x-8)^3$$
   since $R^{(3)}(c) = f^{(3)}(c) - p_2^{(3)}(c) = f^{(3)}(c)$.
4. *If we choose $x = 7.5$, we set $g(x) = \frac{R_2^{(3)}(x)}{3!}(7.5-8)^3$. We want to find c in $(7.5,8)$ such that $R_2(7.5) = g(x)$. We find $c = 7.872817270$.*
5. ***There is a different way to interpret the theorem here. If we let*
   $$F(x) = R_2^{(3)}(x)$$
   and
   $$G(x) = \frac{R_2(x) \cdot 3!}{(x-8)^3}.$$
   We graph $y = F(x)$ and $y = G(x)$ together, we get



6. *So The Largrange Remainder Theorem actually says that if we pick any point x in $(7,9)$ except $x = 8$ for the function G, we can find a c so that $G(x) = F(c)$.*
7. ***Since the green is F, the red is G, we start with any point (x,y) on G (red). Then If $x < 8$, then we can find the corresponding point (by going horizontal direction) on F so that $F(c) = G(x)$.** Similarly, if $x > 8$, we start with any point (x,y) on G, we can find the corresponding point on F so that $F(c) = G(x)$. So if we pick $x = 7.5$, we see $G(7.5) = 0.00150993$ and if we solve $F(x) = 0.00150993$, we get $x = 7.872810231$, which is consistent with the answer we obtained earlier.*
8. *In other words, any point x in $(7,9)$ except $x = 8$, the complex expression $G(x) = \dfrac{R_2(x) \cdot 3!}{(x-8)^3} = \sqrt[3]{x} - \left(\sqrt[3]{8} + \dfrac{(x-8)\sqrt[3]{8}}{24} - \dfrac{(x-8)^2\sqrt[3]{8}}{576}\right)$, can be computed by a simpler expression $F(c) = R_2^{(3)}(c)$ for a proper c.*

I propose the following interpretation.

Theorem **(Yang) Let I be a neighborhood of the point x_0 and let n be a nonnegative integer. Suppose that the function $f : I \to R$ has $n+1$ derivatives. Then for each $x \neq x_0$ in I, **we can find the** point c in (x,x_0) or (x_0,x) such that*

$$G(x) = F(c),$$

where $G(x) = \dfrac{\left(f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\right)(n+1)!}{(x-x_0)^{n+1.}}$ *and* $F(x) = f^{(n+1)}(x)$.

Two Dimension Mean Value Theorem.

Conjecture *Suppose that the function $f : [a,b] \to R$ is continuous and f is differentiable in (a,b). Then for any points $x < y$ in the interval $[a,b]$, we can find x_0 in (x,y) so that*

$$f'(x_o) = \frac{f(y)-f(x)}{y-x}.$$

Example *Click **here** for Maple.*

Example *Consider* $f(x) = x^3 + x^2 - x$ *in the interval* $[-2,2]$.

1. *We define* $F(x) = f'(x)$ *and* $G(x,y) = \dfrac{f(y) - f(x)}{y - x}$.
2. *For one dimensional case, say any two points* $x < y,\ (1, 1.5)$ *in* $(-2,2)$.
   - $G(1, 1.5) = 6.25$ *(slope of the secant line).*
   - *We solve* $F(x) = G(1, 1.5)$ *yields* $x_0 = -1.923232002$ *or* 1.256565336. *In other words,*

$$f'(-1.923232002) = 6.25 \text{ or } f'(1.256565336) = 6.25.$$

3. *We plot the 3d pictures for* $z = F(x)$ *and* $z = G(x,y)$.
4. *We examine that if we pick any points* (x,y) *where* $x < y$ *on* $z = G(x,y)$, *we can always find a corresponding point* (x_0,y) *on* $z = F(x)$ *so that* $F(x_0) = G(x,y)$. *Click **here** for a journal file, and click **here** for Maple file.*
5. *Pick* $x = -1$ *and* $y = 0.2$.
6. *Find the point* $G(-1, 0.2)$ *on the surface.*
7. *Fix* $y = 0.2$ *and find* x_0 *so that* $F(x_0) = G(-1, 0.2)$. *From the journal file above, we see that we have two corresponding* x_0.