For what x is $2^x > x^{10}$?

This problem originated from [T]. With available technology, we shall see how we solve this problem differently.

We first graph the functions 2^x and x^{10} together (with Scientific Workplace or Maple) as follows by selecting proper ranges for x.

$2^x, x^{10}$ $2^{60} = 1.\,1529 \times 10^{18}$

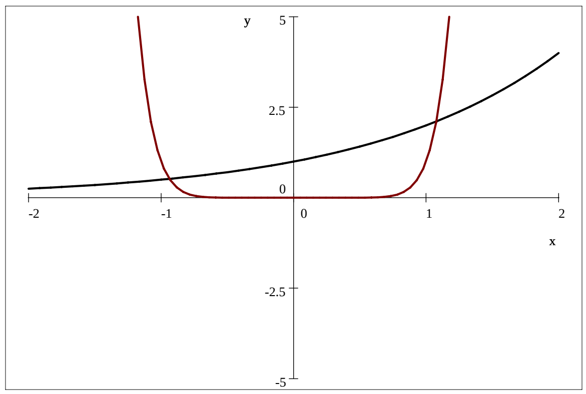

Figure 1

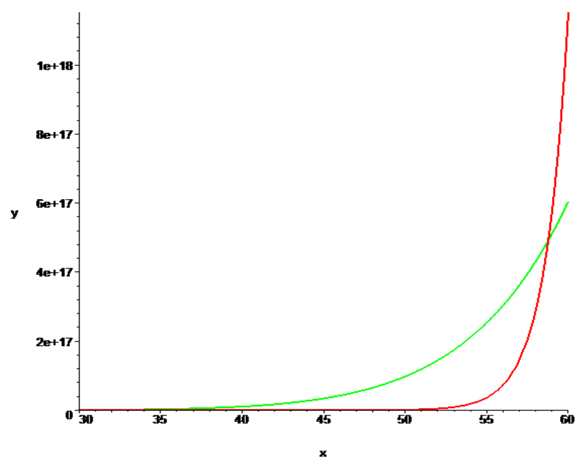

Figure 2

By observing from figures 1 and 2, we predict that there are three solutions for

$$2^x = x^{10}.$$

We shall use **fixed-point** and **Newton's** iterations to find these roots.

Step 1: We set $f(x) = \frac{10}{\ln 2}\ln x$ (by taking "ln" on above equation), and use **fixed-point** iteration (from Scientific Workplace) on f. We shall compute

$$f(x_0),\ f(f(x_0)),\ f(f(f(x_0))),\ldots\ .$$

With the help of Scientific Workplace, and setting the initial value $x_0 = 50$, we obtain Iterates:

$$\begin{array}{c} 50.0 \\ 56.43856 \\ 58.18609 \\ 58.62602 \\ 58.73469 \\ 58.76141 \\ 58.76797 \\ 58.76958 \\ 58.76998 \\ 58.77007 \\ 58.7701 \end{array}$$

Thus the first solution s_1 is approximately 58.7701.

Step 2: We set $g(x) = 2^{0.1x}$ (by taking the 10 root of the equation $2^x = x^{10}$). Use the fixed-point iteration on the function g, and the initial value $x_0 = -0.5$ to obtain the following:

$$\begin{array}{c} -.5 \\ .9659363 \\ 1.069246 \\ 1.07693 \\ 1.077504 \\ 1.077547 \\ 1.07755 \\ 1.07755 \\ 1.07755 \\ 1.07755 \\ 1.07755 \end{array}.$$

Thus the second solution $s_2 \approx 1.07755$.

Step 3: We shall apply Newton's method to obtain the third root. We remark that the fixed-point algorithm fails on this one, see [R].

First we define $h(x) = 2^x - x^{10}$, and if we set

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

we hope $x_n \to$ a zero of f, as $n \to \infty$. In Scientific Workplace, we can define a Newton Iteration function as follows:

$$N(x) = x - \frac{h(x)}{h'(x)}.$$

Now, we apply the iteration on N, with the help from Scientific Workplace, set the initial value to be -2.0. We get the following:

$$\begin{gathered} -2.0 \\ -1.800056 \\ -1.620213 \\ -1.458662 \\ -1.314134 \\ -1.186466 \\ -1.077959 \\ -.9956423 \\ -.9498938 \\ -.937813 \\ -.9371114 \end{gathered}$$

Thus the third solution $s_3 \approx -.9371114$. We conclude that $2^x > x^{10}$ if approximately

$$n \in [58.7701, \infty) \cup (-.9371114, 1.07755)$$

Conclusion

We used computer software Scientic Workplace to solve the preceding problem without using programming. Ten years ago this problem would have been much more difficult using our approach. One might speculate as to how difficult they may be ten years in the future. We are certain that with the help of powerful computer algebra systems, users can discover new methods of solving problems, make intuitive conjectures, which may lead to the discovery of new theorems.

T (bibitem) Travers, For what n is $2^n > n^{10}$? Mathematics Teaching, Harper and Row, 1977, page 146.

R (bibitem) Ralston, A First Course in Numerical Analysis, McGraw-Hill, 1965, page 384, 5(b).