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1. Given that $f(x) = x^2$ for every real number x, simplify the following expressions:

   a. $f[[0,3]]$
      We have
   $$f[[0,3]] = [0,9].$$

   b. $f[(-2,3]]$
      We have
   $$f[[-2,3]] = [0,9].$$

   c. $f^{-1}[[-3,4]]$
      We have
   $$f^{-1}[[-3,4]] = [-2,2].$$

2. Point at the equation $f(x) = x^2$ and then click on the button in your computing toolbar. Then work out the the expressions in parts a and b of the preceding exercise by pointing at them and clicking on the evaluate button.

3. Supply each of the definitions $f(x) = x^2$ and $g(x) = 2 - 3x$ to Scientific Notebook and then ask Scientific Notebook to solve the equation
$$(f \circ g)(x) = (g \circ f)(x).$$

4. Supply the definition
$$f(x) = \frac{x-2}{1-2x}$$
to Scientific Notebook. In this exercise we shall see how to evaluate the composition of the function f with itself up to 20 times starting at a variety of numbers. Open the Maple menu, click on Calculus and move to the right and select Iterate. In the iterate dialogue box fill in the function as f, the starting value as 3 and the number of iterations as 20. Repeat this process with different starting values. Can you draw a conclusion from what you see?

5. Given that $f(x) = x^2$ for all $x \in R$ and $g(x) = 1 + x$ for all $x \in R$, simplify the following expressions:

   a. $(f \circ g)[[0,1]]$
      We have
   $$(f \circ g)[[0,1]] = f[g[[0,1]]] = f[[1,2]] = [1,4].$$

   b. $(g \circ f)[[0,1]]$
      We have
   $$(g \circ f)[[0,1]] = g[f[[0,1]]] = g[[0,1]] = [1,2].$$

   c. $(g \circ g)[[0,1]]$
      We have
   $$(g \circ g)[[0,1]] = g[g[[0,1]]] = g[[1,2]] = [2,3].$$

6. a. Given that $f(x) = (3x-2)/(x+1)$ for all $x \in R \setminus \{-1\}$, determine whether or not f is one-one and find its range.
      Given any number y, the equation
   $$y = \frac{3x-2}{x+1}$$

holds when

$$(x+1)y = 3x-2$$

which can be expressed as

$$x(3-y) = 2+y.$$

In the event that $y = 3$ the lattter equation says that $0 = 5$ and is therefore impossible. If $y \neq 3$ then the equation

$$x(3-y) = 2+y$$

says that

$$x = \frac{2+y}{3-y}.$$

We conclude that the range of f is $R \setminus \{3\}$ and that for every number $y \in R \setminus \{3\}$ there is a unique number x for which $y = f(x)$. Therefore the function f is one-one.

b. Point at the equation

$$y = \frac{3x-2}{x+1}$$

and ask Scientific Notebook to solve for x. How many values of x are given? Is this result consistent with the answer you gave in part a of the question?

7. Suppose that $f : A \to B$ and that $E \subseteq A$. Is it true that $E = f^{-1}[f[E]]$? What if f is one-one? What if f is onto B?
   We certainly have $E \subseteq f^{-1}[f[E]]$ but the inclusion can be strict. For example, if we define $f(x) = x^2$ for every number x then

$$f^{-1}[f[[0,1]]] \neq [0,1].$$

   Now suppose that f is a one-one function from a set A to a set B and that $E \subseteq A$. We shall prove that $f^{-1}[f[E]] \subseteq E$ and, for this purpose, we suppose that $x \in f^{-1}[f[E]]$. We know that $f(x) \in f[E]$ and, using this fact, we choose a member t of the set E such that $f(x) = f(t)$. Since f is one-one we have $x = t$ and so $x \in E$ which is what we needed to show.

8. Suppose that $f : A \to B$ and that $E \subseteq B$. Is it true that $E = f[f^{-1}[E]]$? What if f is one-one? What if f is onto B?
   No it isn't true. There is no reason to suppose that every member of E has to be in the range of f. For example, if we define $f(x) = x^2$ for every number x and $E = [-1,1]$ Then $E \neq f[f^{-1}[E]]$. In the event that f is onto the set B the equation $E = f[f^{-1}[E]]$ will hold.

9. Suppose that $f : A \to B$ and that P and Q are subsets of B. Prove the identities

$$f^{-1}[P \cup Q] = f^{-1}[P] \cup f^{-1}[Q],$$

   Solution: Given any member x of the set A, the condition $x \in f^{-1}[P \cup Q]$ says that $f(x) \in P \cup Q$ which says that either $f(x) \in P$ or $f(x) \in Q$ which says that either $x \in f^{-1}[P]$ or $x \in f^{-1}[Q]$.

$$f^{-1}[P \cap Q] = f^{-1}[P] \cap f^{-1}[Q],$$

$$f^{-1}[P \setminus Q] = f^{-1}[P] \setminus f^{-1}[Q],$$

10. Suppose that $f : A \to B$ and that P and Q are subsets of A. Which of the following statements are true? What if f is one-one? What if f is onto B?

$$f[P \cup Q] = f[P] \cup f[Q]$$

   Hint: This statement is true.

$$f[P \cap Q] = f[P] \cap f[Q]$$

   This statement is false. Give an example. Then prove that the statement is true if f is one-one.

$$f[P \setminus Q] = f[P] \setminus f[Q]$$

This statement is true when f is one-one.

11. Given that f is a one-one function from A to B and that g is a one-one function from B to C, prove that the function $g \circ f$ is one-one from A to C.

Solution: We need to prove that whenever t and x are members of the set A and $t \neq x$ we have

$$(g \circ f)(t) \neq (g \circ f)(x).$$

Suppose that t and x are members of the set A and that $t \neq x$. Since f is one-one we have $f(t) \neq f(x)$. Therefore, since g is one-one we have $g(f(t)) \neq g(f(x))$ and we have shown that $(g \circ f)(t) \neq (g \circ f)(x)$.

12. Given that f is a function from A onto B and that g is a function from B onto C, prove that the function $g \circ f$ is a function from A onto C.
Suppose that $z \in C$. Using the fact that the function g is onto the set C, choose a member y of the set B such that $z = g(y)$. Now, using the fact that the function f is onto the set B we choose a member x of the set A such that $y = f(x)$. We have found a member x of A such that $z = (f \circ g)(x)$. Therefore C is the range of the function $f \circ g$.

13. Given that $f : A \to B$ and that $g : B \to C$ and that the function $g \circ f$ is one-one, prove that f must be one-one. Give an example to show that the function g does not have to be one-one.

Solution: To prove that f is one-one, suppose that x_1 and x_2 are members of the set A and that $x_1 \neq x_2$. Since the function $g \circ f$ is one-one we know that $g(f(x_1)) \neq g(f(x_2))$ and we see at once that $f(x_1) \neq f(x_2)$. Now we construct an example to show that the function g does not have to be one-one. We define $f(x) = x$ for every $x \in [0,1]$ and we define

$$g(x) = \begin{cases} x & \text{if} \quad x \in [0,1] \\ 2 & \text{if} \quad 1 < x \leq 5 \end{cases}$$

14. Given that f is a function from A onto B and that $g : B \to C$ and that the function $g \circ f$ is one-one, prove that both of the functions f and g have to be one-one.

Solution: To see that f is one-one, suppose that x and t are members of the set A and that $t \neq x$. Since $g(f(t)) \neq g(f(x))$ we see at once that $f(t) \neq f(x)$.

Now to see that g is one-one, suppose that u and y are members of the set B and that $u \neq y$. Using the fact that the function f is onto the set B we choose members t and x of A such that $u = f(t)$ and $y = f(x)$. We see at once that $t \neq x$ and therefore

$$g(u) = g(f(t)) \neq g(f(x)) = g(y).$$

15. Given any set S, the identity function i_S on S is defined by $i_S(x) = x$ for every $x \in S$. Prove that if f is a one-one function from a set A onto a set B then $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.
There is really nothing to prove. The fact that

$$f^{-1}(f(x)) = x = i_A(x)$$

for every $x \in A$ follows at once from the definition of the function f^{-1}. The equation $f \circ f^{-1} = i_B$ follows in a similar manner.

16. Suppose that $f : A \to B$.

    a. Given that there exists a function $g : B \to A$ such that $g \circ f = i_A$, what can be said about the functions f and g?
    The function f must be one-one because if t and x belong to A and $f(t) = f(x)$ then we have

    $$t = g(f(t)) = g(f(x)) = x.$$

    The function g must be onto the set A because if x is any member of A we have

    $$x = g(f(x)).$$

    b. Given that there exists a function $h : B \to A$ such that $f \circ h = i_B$, what can be said about the functions f and h?

The function f must be onto the set B and the function h must be one-one.

c. Given that there exists a function $g : B \to A$ such that $g \circ f = i_A$ and that there exists a function $h : B \to A$ such that $f \circ h = i_B$, what can be said about the functions f, g and h?
From parts a and b we see that all three functions are one-one and that f is onto B and that the functions g and h are onto A.

17. As in a previous example, we define

$$f_a(x) = \frac{x-a}{1-ax}$$

whenever $a \in (-1,1)$ and $x \in [-1,1]$.

a. Prove that if a and b belong to $(-1,1)$ then so does the number

$$c = \frac{a+b}{1+ab}.$$

Hint: An quick way to do this exercise is to observe that $c = f_{-b}(a)$.
There really isn't much more to say here. The earlier material showed that f_{-b} is a one-one function from $[0,1]$ onto $[0,1]$ and we see at once that $f_{-b}(-1) = -1$ and $f_{-b}(1) = 1$.

b. Given a and b in $(-1,1)$ and

$$c = \frac{a+b}{1+ab},$$

prove that $f_b \circ f_a = f_c$.
Given any number $x \in [0,1]$ we have

$$f_b \circ f_a(x) = f_b(f_a(x)) = f_b\left(\frac{x-a}{1-ax}\right) = \frac{\frac{x-a}{1-ax} - b}{1 - b\left(\frac{x-a}{1-ax}\right)}$$

$$= \frac{x-a-b(1-ax)}{1-ax-b(x-a)} = \frac{x(1+ab)-a-b}{1+ab-(a+b)x}$$

$$= \frac{x - \frac{a+b}{1+ab}}{\frac{1+ab}{a+b} - x} = f_c(x)$$