

1. Prove the following assertions:

- a. If a, b, c, x and y are positive numbers and $a^x = b$ and $b^y = c$, then $a^{xy} = c$.

The point of this exercise is to emphasize that you don't already have numbers a, b, c, x and y quantified for you in the statement of the exercise.

Solution: You need to begin: Suppose that a, b, c, x and y are positive numbers and that $a^x = b$ and $b^y = c$. Then you can argue that

$$a^{xy} = (a^x)^y = b^y = c.$$

- b. If two integers m and n are both even then mn has a factor 4.

Solution: You need to begin: Suppose that m and n are even integers. Since m is even we know that $m/2$ is an integer and since n is even we know that $n/2$ is an integer. Since

$$mn = 4\left(\frac{m}{2}\right)\left(\frac{n}{2}\right)$$

we conclude that mn has a factor 4.

- c. If an integer m is even and an integer n has a factor 3 then mn has a factor 6.

You need to begin: Suppose that m and n are integers and that m is even and that n has a factor 3. We know that the numbers $m/2$ and $n/3$ are both integers. Since

$$mn = 6\left(\frac{m}{2}\right)\left(\frac{n}{3}\right)$$

we deduce that mn has a factor 6.

2. "Ladies and gentlemen of the jury" said the prosecutor, "We shall demonstrate beyond a shadow of doubt that on the night of June 13, 1997, the accused, Slippery Sam Carlisle, did willfully, unlawfully and maliciously kill and murder the deceased, Archibald Bott by striking him on the head with a blunt instrument". Outline a strategy that the prosecutor might use in order to prove this charge. How many separate assertions must the prosecutor prove in order to carry out his promise to the jury?

Of course this problem isn't serious in suggesting that Slippery Sam should be found not guilty if any one of the actions described by the prosecutor turns out to be untrue. But, had the prosecutor been a mathematician he would have agreed that his obligation, in order to convict Slippery Sam, is to show that Sam's action was wilful and also that it was unlawful and also that it was malicious and also that it involved killing Mr. Bott and also that it involved murdering Mr. Bott and also that the act was performed by striking Mr. Bott on the head and also that the murder weapon was blunt. If even one of these conditions is found to be false then a mathematician might be inclined to find Slippery Sam not guilty.

3. One of the basic laws of arithmetic tells us that if a and b are any two numbers satisfying the condition $a < b$ and if $x > 0$ then $ax < bx$. Show how this law may be used to show that if $0 < u < 1$ and $0 < v < 1$ then $0 < uv < 1$.

Solution: We begin the proof by quantifying u and v : Suppose that u and v are numbers satisfying the inequalities $0 < u < 1$ and $0 < v < 1$. Since $u < 1$ and $v > 0$ we know that $uv < 1v$ which tells us that $uv < v$. Since $uv < v$ and $v < 1$ we conclude that $uv < 1$.

Now since $0 < v$ and $u > 0$ we know that $0u < uv$ which tells us that $0 < uv$. Therefore $0 < uv < 1$.

4. In this exercise, if we are given three nonnegative integers a, b and c then the integer that consists of a hundreds, b tens and c units will be written as $\lceil a, b, c \rceil$. Given nonnegative integers a, b and c , prove the assertion $P \wedge Q \wedge R \wedge S$ where P, Q, R and S are, respectively, the following assertions

P . If the number $\lceil a, b, c \rceil$ is divisible by 3 then the number $a + b + c$ is also divisible by 3.

Q . If the number $a + b + c$ is divisible by 3 then the number $\lceil a, b, c \rceil$ is also divisible by 3.

R . If the number $\lceil a, b, c \rceil$ is divisible by 9 then the number $a + b + c$ is also divisible by 9.

S . If the number $a + b + c$ is divisible by 9 then the number $\lceil a, b, c \rceil$ is also divisible by 9.

Hint: In this exercise we are actually given three nonnegative integers a, b and c and so there is no need

to quantify them. In order to prove the assertion $P \wedge Q \wedge R \wedge S$ we have to show that all of the statements P , Q , R , and S are true. We therefore have four separate proofs to write. We show the proof of statement Q here and leave the rest to you:

Suppose that the number $a + b + c$ is divisible by 3. We need to show that the number

$$\lceil a, b, c \rceil = 100a + 10b + c$$

is also divisible by 3. Now since the number $\frac{a+b+c}{3}$ is an integer and since

$$\begin{aligned}\lceil a, b, c \rceil &= 100a + 10b + c = 99a + 9b + a + b + c \\ &= 3\left(33a + 3b + \frac{a+b+c}{3}\right)\end{aligned}$$

we know that $\lceil a, b, c \rceil$ has a factor 3.