1. Suppose that 
\[ S = [0, 1) \cup (1, 2). \]

a. What is the set of interior points of \( S \)?
The set of interior points of \( S \) is \( (0, 1) \cup (1, 2). \)

b. Given that \( U \) is the set of interior points of \( S \), evaluate \( U \).
\[ (0, 1) \cup (1, 2) = [0, 1] \cup [1, 2] = \mathbb{R}. \]
The purpose of parts a and b is to exhibit a set \( S \) such that, if \( U \) is the set of interior points of \( S \) then \( U = \mathbb{R} \).

c. Give an example of a set \( S \) of real numbers such that if \( U \) is the set of interior points of \( S \) then \( U \neq [0, 1] \).

We could take \( S \) to be a singleton like \( \{3\} \) or it could be the set of all integers. It could also be the set of all rational numbers between 0 and 1.

d. Give an example of a subset \( S \) of the interval \( [0, 1] \) such that \( S = [0, 1] \) but if \( U \) is the set of interior points of \( S \) then \( U \neq [0, 1] \).

Once again, take the set of all rational numbers between 0 and 1.

2. Given that 
\[ S = \left\{ \frac{1}{n} \mid n \in \mathbb{Z}^+ \right\}, \]
evaluate \( S \).

Hint: Show that 
\[ S = \{0\} \cup \left\{ \frac{1}{n} \mid n \in \mathbb{Z}^+ \right\}. \]

First show that \( 0 \in S \). Then observe that every negative number belongs to the set \((-\infty, 0)\) and that if \( x \) is any positive number then \( x \) belongs to the interval
\[ \left( \frac{1}{n+1}, \frac{1}{n} \right) \]
for some positive integer \( n \).

3. Given that \( S \) is a set of real numbers, that \( H \) is a closed set and that \( S \subseteq H \), prove that \( \overline{S} \subseteq \overline{H} \).

We could argue that \( \overline{S} \subseteq \overline{H} \) and that, because \( H \) is closed, \( \overline{H} = H \).

4. Given two sets \( A \) and \( B \) of real numbers, prove that 
\[ \overline{A} \cup \overline{B} = \overline{A \cup B}. \]

Solution: Since \( A \subseteq \overline{A} \) and \( B \subseteq \overline{B} \) we have 
\[ A \cup B \subseteq \overline{A} \cup \overline{B}. \]

and therefore, since the union of the two closed sets \( \overline{A} \) and \( \overline{B} \) is closed we have 
\[ \overline{A} \cup \overline{B} \subseteq \overline{A} \cup \overline{B}. \]

On the other hand, since \( A \) is included in the closed set \( \overline{A} \cup \overline{B} \) we have 
\[ \overline{A} \subseteq \overline{A} \cup \overline{B}. \]

and, similarly we can see that
\[ \overline{B} \subseteq \overline{A} \cup \overline{B}. \]

and so
\[ \overline{A} \cup \overline{B} \subseteq \overline{A} \cup \overline{B}. \]

Therefore
\[ \overline{A} \cup \overline{B} = \overline{A \cup B}. \]

5. Given two sets \( A \) and \( B \) of real numbers, prove that
\[ \mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A} \cap \mathcal{B}. \]

Do the two sides of this inclusion have to be equal? What if \( \mathcal{A} \) and \( \mathcal{B} \) are open? What if they are closed?

Since \( \mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A} \) we have \( \mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A} \) and similarly that \( \mathcal{A} \cap \mathcal{B} \subseteq \mathcal{B} \). Thus
\[ \mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A} \cap \mathcal{B}. \]

Now observe that if \( A = (0, 1) \) and \( B = (1, 2) \) then
\[ \mathcal{A} \cap \mathcal{B} = [0, 1] \cap [1, 2] = \{1\} \]
and
\[ \mathcal{A} \cap \mathcal{B} = \emptyset = \emptyset. \]

Of course, we could give more spectacular examples like \( \mathcal{A} = Q \) and \( \mathcal{B} = R \setminus Q \).

6. Prove that if \( S \) is any set of real numbers then the set \( R \setminus S \) is the set of interior points of the set \( R \setminus S \).

Most students should be encouraged to write two separate arguments here. The first task is to show that every member of the set \( R \setminus S \) must be an interior point of \( R \setminus S \). Then one should show that every interior point of \( R \setminus S \) must belong to \( R \setminus S \).

On the other hand, a strong student could be permitted to observe that if \( x \) is any given number then the statement that \( x \) does not belong to \( \overline{S} \) is the statement that there exists a number \( \delta > 0 \) such that \( (x - \delta, x + \delta) \cap S = \emptyset \), and that the latter equation is just the condition that \( (x - \delta, x + \delta) \subseteq R \setminus S \).

7. Given that \( a \) is an upper bound of a given set \( S \) of real numbers, prove that the following two conditions are equivalent:
   a. We have \( a = \sup S \).
   b. We have \( a \in \overline{S} \).

To prove that condition a implies condition b we assume that \( a = \sup S \). We need to show that \( a \in \overline{S} \).

Suppose that \( \delta > 0 \). Using the fact that \( a \) is the least upper bound of \( S \) and that \( a - \delta < a \) we choose a member \( x \) of \( S \) such that \( a - \delta < x \). Since \( x \in (a - \delta, a + \delta) \cap S \) we have \( (a - \delta, a + \delta) \cap S \neq \emptyset \).

To prove that condition b implies condition a we assume that \( a \in \overline{S} \). We need to show that \( a \) is the least upper bound of \( S \). Suppose that \( p < a \). Since the set \( (p, \infty) \) is a neighborhood of \( a \) we have \( (p, \infty) \cap S \neq \emptyset \).

Thus, since \( a \) is an upper bound of \( S \) and since no number \( p < a \) can be an upper bound of \( S \) we conclude that \( a \) is the least upper bound of \( S \).

8. Is it true that if \( A \) and \( B \) are sets of real numbers and

\[ \mathcal{A} = \mathcal{B} = \mathbb{R} \]

then \( \mathcal{A} \cap \mathcal{B} = \mathbb{R} \)?

The answer is no. Look at \( A = Q \) and \( B = R \setminus Q \).

9. Prove that if \( A \) and \( B \) are open sets and

\[ \mathcal{A} = \mathcal{B} = \mathbb{R} \]

then \( \mathcal{A} \cap \mathcal{B} = \mathbb{R} \). What if only one of the sets \( A \) and \( B \) is open?

Solution: All we need to know is that at least one of the sets \( A \) and \( B \) is open. Suppose that \( A \) and \( B \) are sets of real numbers, that
\[ \mathcal{A} = \mathcal{B} = \mathbb{R} \]
and that the set \( A \) is open.

To prove that
\[ \mathcal{A} \cap \mathcal{B} = \mathbb{R}, \]
suppose that \( x \) is any real number and that \( \delta > 0 \). Since \( x \in \mathcal{A} \) we know that the set
\[ (x - \delta, x + \delta) \cap A \]
is nonempty and we also know that this set is open. Therefore, since \( \mathcal{B} = \mathbb{R} \) we know that
\[ (x - \delta, x + \delta) \cap A \cap B \neq \emptyset. \]

We have therefore shown that every real number must belong to \( \mathcal{A} \cap \mathcal{B} \).
10. Two sets $A$ and $B$ are said to be separated from each other if
\[ \overline{A} \cap B = A \cap \overline{B} = \emptyset. \]
Which of the following pairs of sets are separated from each other?

- a. $[0,1]$ and $[2,3]$. Yes.
- b. $(0,1)$ and $(1,2)$. Yes.
- c. $(0,1)$ and $(1,2)$. No because $(0,1) \cap (1,2) = \{1\} \neq \emptyset$.
- d. $Q$ and $R \setminus Q$. No.

11. Prove that if $A$ and $B$ are closed and disjoint from one another then $A$ and $B$ are separated from each other. Suppose that $A$ and $B$ are closed and disjoint from one another. Since $A = \overline{A}$ and $B = \overline{B}$, the fact that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ follows at once from the fact that $A \cap B = \emptyset$.

12. Prove that if $A$ and $B$ are open and disjoint from one another then $A$ and $B$ are separated from each other. Suppose that $A$ and $B$ are open and disjoint from one another. Given any number $x \in A$, we deduce from the fact that $A$ is a neighborhood of $x$ and $A \cap B = \emptyset$ that $x$ is not close to $B$. Therefore $A \cap \overline{B} = \emptyset$ and we see similarly that $\overline{A} \cap B = \emptyset$.

13. Suppose that $S$ is a set of real numbers. Prove that the two sets $S$ and $R \setminus S$ will be separated from each other if and only if the set $S$ is both open and closed. What then do we know about the sets $S$ for which $S$ and $R \setminus S$ are separated from each other?

Suppose that $S$ and $R \setminus S$ are separated from each other. To show that $S$ is open, suppose that $x \in S$. Since $S \cap (R \setminus S) = \emptyset$ we know that $x$ is not close to $R \setminus S$. Choose $\delta > 0$ such that
\[ (x-\delta,x+\delta) \cap (R \setminus S) = \emptyset \]
and observe that $(x-\delta,x+\delta) \subseteq S$. Thus $S$ is open and a similar argument shows that $R \setminus S$ is also open. We therefore know that if the sets $S$ and $R \setminus S$ are separated from one another then $S$ is both open and closed.

Now suppose that $S$ is both open and closed. Since the two set $S$ and $R \setminus S$ are closed and disjoint from one other they are separated from one another.

14. This exercise refers to the notion of a subgroup of $R$ that was introduced in an earlier exercise. That exercise should be completed before you start this one.

- a. Given that $H$ and $K$ are subgroups of $R$, prove that the set $H + K$ defined in the sense of an earlier exercise is also a subgroup of $R$.
   To prove that $H + K$ is a subgroup of $R$ we need to show that $H + K$ is nonempty and that the sum and difference of any members of $H + K$ must always belong to $H + K$.
   To show that $H + K$ is nonempty we use the fact that $H$ and $K$ are nonempty to choose $x \in H$ and $y \in K$. Since $x + y \in H + K$ we have $H + K \neq \emptyset$.
   Now suppose that $w_1$ and $w_2$ are any members of the set $H + K$. Choose members $x_1$ and $x_2$ of $H$ and members $y_1$ and $y_2$ of $K$ such that $w_1 = x_1 + y_1$ and $w_2 = x_2 + y_2$. Since the numbers $x_1 + x_1$ and $x_1 - x_2$ belong to $H$ and the numbers $y_1 + y_2$ and $y_1 - y_2$ belong to $K$, and since
   \[ w_1 + w_2 = (x_1 + x_2) + (y_1 + y_2) \]
   and
   \[ w_1 - w_2 = (x_1 - x_2) + (y_1 - y_2) \]
   we see at once that $w_1 + w_2$ and $w_1 - w_2$ belong to $H + K$.

- b. Prove that if $a$, $b$ and $c$ are integers and if
   \[ a\sqrt{2} = b\sqrt{3} + c \]
   then $a = b = c = 0$.
   Solution: From the equation
   \[ a\sqrt{2} = b\sqrt{3} + c \]
   we see that
\[2a^2 = 3b^2 + 2bc\sqrt{3} + c^2.\]

Therefore, unless \(bc = 0\) we have
\[\sqrt{3} = \frac{2a^2 - 3b^2 - c^2}{2bc},\]
which contradicts the fact that the number \(\sqrt{3}\) is irrational. Therefore at least one of the number \(b\) and \(c\) must be zero.

In the event that \(c = 0\), the equation
\[a\sqrt{2} = b\sqrt{3} + c\]
becomes
\[a\sqrt{2} = b\sqrt{3}\]
and, unless \(a = 0\), the latter equation gives us
\[\frac{\sqrt{2}}{\sqrt{3}} = \frac{b}{a}\]
which contradicts the fact that \(\sqrt{2}/\sqrt{3}\) is irrational. So in the case \(c = 0\) we also have \(a = 0\) and we see at once that \(b = 0\) as well.

In the event that \(b = 0\), the equation
\[a\sqrt{2} = b\sqrt{3} + c\]
becomes
\[a\sqrt{2} = c\]
and, unless \(a = 0\), the latter equation gives us
\[\sqrt{2} = \frac{c}{a}\]
which contradicts the irrationality of \(\sqrt{2}\). So, once again, \(a = 0\) and we see at once that \(c = 0\) as well.

**c.** Prove that if \(m, n, p\) and \(q\) are integers then it is impossible to have
\[\frac{\sqrt{2} - m}{n} = \frac{\sqrt{3} - p}{q}\]
and deduce that if \(\alpha\) is any real number and if \(H = \langle n\alpha \mid n \in \mathbb{Z} \rangle\) then the subgroup \(H + \mathbb{Z}\) cannot contain both of the numbers \(\sqrt{2}\) and \(\sqrt{3}\).

**Solution:** The equation
\[\frac{\sqrt{2} - m}{n} = \frac{\sqrt{3} - p}{q}\]
implies that
\[q\sqrt{2} = n\sqrt{3} - np + mq\]
which, by part b, tells us that
\[0 = q = n = mq - np\]
which is clearly impossible since \(n\) and \(q\) appear denominators of the fractions in the equation
\[\frac{\sqrt{2} - m}{n} = \frac{\sqrt{3} - p}{q}\]

Now, to obtain a contradiction, suppose that the subgroup \(H + \mathbb{Z}\) contains both of the numbers \(\sqrt{2}\) and \(\sqrt{3}\). Choose integers \(m\) and \(n\) such that
\[\sqrt{2} = m + na\]
and choose integers \(p\) and \(q\) such that
\[\sqrt{3} = p + qa.\]

Since \(\sqrt{2}\) is irrational, we know that \(\sqrt{2} \neq m\) and so \(n \neq 0\); and we know similarly that \(q \neq 0\). Thus
\[\frac{\sqrt{2} - m}{n} = \frac{\sqrt{3} - p}{q}\]
which we know to be impossible.

d. Suppose that $G$ is a subgroup of $R$ other than $\langle 0 \rangle$, that

$$p = \inf \{x \in G \mid x > 0\}$$

and that the number $p$ is positive. Prove that the set $G$ is closed.
Solution: We know from an earlier exercise that

$$G = \langle np \mid n \in \mathbb{Z} \rangle.$$ 

e. Prove that if $G$ is a subgroup of $R$ other than $\langle 0 \rangle$ and that $G$ has no least positive member then $\overline{G} = R$.
Solution: This fact was established in an earlier exercise.

f. Suppose that $\alpha$ is an irrational number, that

$$H = \langle n\alpha \mid n \in \mathbb{Z} \rangle$$

and that $G = H + Z$ (in the sense of this exercise). Prove that although the sets $H$ and $Z$ are closed subgroups of $R$ and although the set $G$ is also a subgroup of $R$, the set $G$ is not closed.
Solution: Since $G$ cannot contain both of the numbers $\sqrt{2}$ and $\sqrt{3}$ we know that $G \neq R$. To show that $G$ is not closed we shall make the observation that $\overline{G} = R$ and, for this purpose, all we have to show is that if

$$p = \inf \{x \in G \mid x > 0\}$$

then $p = 0$. Suppose that $p$ is defined in this way and, to obtain a contradiction, suppose that $p > 0$. We know that

$$G = \langle np \mid n \in \mathbb{Z} \rangle$$

and, using the fact that both of the numbers 1 and $\alpha$ belong to $G$, we choose integers $m$ and $n$ such that

$$1 = mp$$

and

$$\alpha = np.$$ 

From the fact that $p = 1/m$ we see that $p$ is rational but from the fact that $p = a/n$ we see that $p$ must be irrational. Thus we have arrived at the promised contradiction.