

1. Suppose that

$$S = [0, 1) \cup (1, 2).$$

- a. What is the set of interior points of  $S$ ?  
The set of interior points of  $S$  is  $(0, 1) \cup (1, 2)$ .

- b. Given that  $U$  is the set of interior points of  $S$ , evaluate  $\overline{U}$ .

$$\overline{(0, 1) \cup (1, 2)} = [0, 1] \cup [1, 2] = \overline{S}.$$

The purpose of parts a and b is to exhibit a set  $S$  such that, if  $U$  is the set of interior points of  $S$  then  $\overline{U} = \overline{S}$ .

- c. Give an example of a set  $S$  of real numbers such that if  $U$  is the set of interior points of  $S$  then  $\overline{U} \neq \overline{S}$ .  
We could take  $S$  to be a singleton like  $\{3\}$  or it could be the set of all integers. It could also be the set of all rational numbers between 0 and 1.
- d. Give an example of a subset  $S$  of the interval  $[0, 1]$  such that  $\overline{S} = [0, 1]$  but if  $U$  is the set of interior points of  $S$  then  $\overline{U} \neq [0, 1]$ .  
Once again, take the set of all rational numbers between 0 and 1.

2. Given that

$$S = \left\{ \frac{1}{n} \mid n \in \mathbf{Z}^+ \right\},$$

evaluate  $\overline{S}$ .

Hint: Show that

$$\overline{S} = \{0\} \cup \left\{ \frac{1}{n} \mid n \in \mathbf{Z}^+ \right\}.$$

First show that  $0 \in \overline{S}$ . Then observe that every negative number belongs to the set  $(-\infty, 0)$  and that if  $x$  is any positive number then  $x$  belongs to the interval

$$\left( \frac{1}{n+1}, \frac{1}{n} \right)$$

for some positive integer  $n$ .

3. Given that  $S$  is a set of real numbers, that  $H$  is a closed set and that  $S \subseteq H$ , prove that  $\overline{S} \subseteq H$ .  
We could argue that  $\overline{S} \subseteq \overline{H}$  and that, because  $H$  is closed,  $\overline{H} = H$ .
4. Given two sets  $A$  and  $B$  of real numbers, prove that

$$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

Solution: Since  $A \subseteq \overline{A}$  and  $B \subseteq \overline{B}$  we have

$$A \cup B \subseteq \overline{A} \cup \overline{B}.$$

and therefore, since the union of the two closed sets  $\overline{A}$  and  $\overline{B}$  is closed we have

$$\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}.$$

On the other hand, since  $A$  is included in the closed set  $\overline{A \cup B}$  we have

$$\overline{A} \subseteq \overline{A \cup B}$$

and, similarly we can see that

$$\overline{B} \subseteq \overline{A \cup B}$$

and so

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}.$$

Therefore

$$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

5. Given two sets  $A$  and  $B$  of real numbers, prove that

$$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}.$$

Do the two sides of this inclusion have to be equal? What if  $A$  and  $B$  are open? What if they are closed? Since  $A \cap B \subseteq A$  we have  $\overline{A \cap B} \subseteq \overline{A}$  and similarly that  $\overline{A \cap B} \subseteq \overline{B}$ . Thus

$$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}.$$

Now observe that if  $A = (0, 1)$  and  $B = (1, 2)$  then

$$\overline{A \cap B} = [0, 1] \cap [1, 2] = \{1\}$$

and

$$\overline{A} \cap \overline{B} = \overline{\emptyset} = \emptyset.$$

Of course, we could give more spectacular examples like  $A = Q$  and  $B = R \setminus Q$ .

6. Prove that if  $S$  is any set of real numbers then the set  $R \setminus \overline{S}$  is the set of interior points of the set  $R \setminus S$ . Most students should be encouraged to write two separate arguments here. The first task is to show that every member of the set  $R \setminus \overline{S}$  must be an interior point of  $R \setminus S$ . Then one should show that every interior point of  $R \setminus S$  must belong to  $R \setminus \overline{S}$ .

On the other hand, a strong student could be permitted to observe that if  $x$  is any given number then the statement that  $x$  does not belong to  $\overline{S}$  is the statement that there exists a number  $\delta > 0$  such that  $(x - \delta, x + \delta) \cap S = \emptyset$ , and that the latter equation is just the condition that  $(x - \delta, x + \delta) \subseteq R \setminus S$ .

7. Given that  $\alpha$  is an upper bound of a given set  $S$  of real numbers, prove that the following two conditions are equivalent:
- We have  $\alpha = \sup S$ .
  - We have  $\alpha \in \overline{S}$ .

To prove that condition a implies condition b we assume that  $\alpha = \sup S$ . We need to show that  $\alpha \in \overline{S}$ .

Suppose that  $\delta > 0$ . Using the fact that  $\alpha$  is the least upper bound of  $S$  and that  $\alpha - \delta < \alpha$  we choose a member  $x$  of  $S$  such that  $\alpha - \delta < x$ . Since  $x \in (\alpha - \delta, \alpha + \delta) \cap S$  we have  $(\alpha - \delta, \alpha + \delta) \cap S \neq \emptyset$ .

To prove that condition b implies condition a we assume that  $\alpha \in \overline{S}$ . We need to show that  $\alpha$  is the least upper bound of  $S$ . Suppose that  $p < \alpha$ . Since the set  $(p, \infty)$  is a neighborhood of  $\alpha$  we have  $(p, \infty) \cap S \neq \emptyset$ . Thus, since  $\alpha$  is an upper bound of  $S$  and since no number  $p < \alpha$  can be an upper bound of  $S$  we conclude that  $\alpha$  is the least upper bound of  $S$ .

8. Is it true that if  $A$  and  $B$  are sets of real numbers and

$$\overline{A} = \overline{B} = \mathbf{R}$$

then  $\overline{A \cap B} = R$ ?

The answer is no. Look at  $A = Q$  and  $B = R \setminus Q$ .

9. Prove that if  $A$  and  $B$  are open sets and

$$\overline{A} = \overline{B} = \mathbf{R}$$

then  $\overline{A \cap B} = R$ . What if only one of the sets  $A$  and  $B$  is open?

Solution: All we need to know is that at least one of the sets  $A$  and  $B$  is open. Suppose that  $A$  and  $B$  are sets of real numbers, that

$$\overline{A} = \overline{B} = \mathbf{R}$$

and that the set  $A$  is open.

To prove that

$$\overline{A \cap B} = \mathbf{R},$$

suppose that  $x$  is any real number and that  $\delta > 0$ . Since  $x \in \overline{A}$  we know that the set

$$(x - \delta, x + \delta) \cap A$$

is nonempty and we also know that this set is open. Therefore, since  $\overline{B} = R$  we know that

$$(x - \delta, x + \delta) \cap A \cap B \neq \emptyset.$$

We have therefore shown that every real number must belong to  $\overline{A \cap B}$ .

10. Two sets  $A$  and  $B$  are said to be separated from each other if

$$\overline{A} \cap B = A \cap \overline{B} = \emptyset.$$

Which of the following pairs of sets are separated from each other?

- $[0, 1]$  and  $[2, 3]$ . Yes.
  - $(0, 1)$  and  $(1, 2)$ . Yes.
  - $(0, 1]$  and  $(1, 2)$ . No because  $(0, 1] \cap \overline{(1, 2)} = \{1\} \neq \emptyset$ .
  - $Q$  and  $R \setminus Q$ . No.
11. Prove that if  $A$  and  $B$  are closed and disjoint from one another then  $A$  and  $B$  are separated from each other. Suppose that  $A$  and  $B$  are closed and disjoint from one another. Since  $A = \overline{A}$  and  $B = \overline{B}$ , the fact that  $\overline{A} \cap B = A \cap \overline{B} = \emptyset$  follows at once from the fact that  $A \cap B = \emptyset$ .
12. Prove that if  $A$  and  $B$  are open and disjoint from one another then  $A$  and  $B$  are separated from each other. Suppose that  $A$  and  $B$  are open and disjoint from one another. Given any number  $x \in A$ , we deduce from the fact that  $A$  is a neighborhood of  $x$  and  $A \cap B = \emptyset$  that  $x$  is not close to  $B$ . Therefore  $A \cap \overline{B} = \emptyset$  and we see similarly that  $\overline{A} \cap B = \emptyset$ .
13. Suppose that  $S$  is a set of real numbers. Prove that the two sets  $S$  and  $R \setminus S$  will be separated from each other if and only if the set  $S$  is both open and closed. What then do we know about the sets  $S$  for which  $S$  and  $R \setminus S$  are separated from each other? Suppose that  $S$  and  $R \setminus S$  are separated from each other. To show that  $S$  is open, suppose that  $x \in S$ . Since  $S \cap (\overline{R \setminus S}) = \emptyset$  we know that  $x$  is not close to  $R \setminus S$ . Choose  $\delta > 0$  such that

$$(x - \delta, x + \delta) \cap (R \setminus S) = \emptyset$$

and observe that  $(x - \delta, x + \delta) \subseteq S$ . Thus  $S$  is open and a similar argument shows that  $R \setminus S$  is also open. We therefore know that if the sets  $S$  and  $R \setminus S$  are separated from one another then  $S$  is both open and closed. Now suppose that  $S$  is both open and closed. Since the two sets  $S$  and  $R \setminus S$  are closed and disjoint from one another they are separated from one another.

14. This exercise refers to the notion of a subgroup of  $R$  that was introduced in an earlier exercise. That exercise should be completed before you start this one.
- Given that  $H$  and  $K$  are subgroups of  $R$ , prove that the set  $H + K$  defined in the sense of an earlier exercise is also a subgroup of  $R$ .  
To prove that  $H + K$  is a subgroup of  $R$  we need to show that  $H + K$  is nonempty and that the sum and difference of any members of  $H + K$  must always belong to  $H + K$ .  
To show that  $H + K$  is nonempty we use the fact that  $H$  and  $K$  are nonempty to choose  $x \in H$  and  $y \in K$ . Since  $x + y \in H + K$  we have  $H + K \neq \emptyset$ .  
Now suppose that  $w_1$  and  $w_2$  are any members of the set  $H + K$ . Choose members  $x_1$  and  $x_2$  of  $H$  and members  $y_1$  and  $y_2$  of  $K$  such that  $w_1 = x_1 + y_1$  and  $w_2 = x_2 + y_2$ . Since the numbers  $x_1 + x_1$  and  $x_1 - x_2$  belong to  $H$  and the numbers  $y_1 + y_2$  and  $y_1 - y_2$  belong to  $K$ , and since

$$w_1 + w_2 = (x_1 + x_2) + (y_1 + y_2)$$

and

$$w_1 - w_2 = (x_1 - x_2) + (y_1 - y_2)$$

we see at once that  $w_1 + w_2$  and  $w_1 - w_2$  belong to  $H + K$ .

- Prove that if  $a, b$  and  $c$  are integers and if

$$a\sqrt{2} = b\sqrt{3} + c$$

then  $a = b = c = 0$ .

Solution: From the equation

$$a\sqrt{2} = b\sqrt{3} + c$$

we see that

$$2a^2 = 3b^2 + 2bc\sqrt{3} + c^2.$$

Therefore, unless  $bc = 0$  we have

$$\sqrt{3} = \frac{2a^2 - 3b^2 - c^2}{2bc}$$

which contradicts the fact that the number  $\sqrt{3}$  is irrational. Therefore at least one of the number  $b$  and  $c$  must be zero.

In the event that  $c = 0$ , the equation

$$a\sqrt{2} = b\sqrt{3} + c$$

becomes

$$a\sqrt{2} = b\sqrt{3}$$

and, unless  $a = 0$ , the latter equation gives us

$$\sqrt{\frac{2}{3}} = \frac{b}{a}$$

which contradicts the fact that  $\sqrt{\frac{2}{3}}$  is irrational. So in the case  $c = 0$  we also have  $a = 0$  and we see at once that  $b = 0$  as well.

In the event that  $b = 0$ , the equation

$$a\sqrt{2} = b\sqrt{3} + c$$

becomes

$$a\sqrt{2} = c$$

and, unless  $a = 0$ , the latter equation gives us

$$\sqrt{2} = \frac{c}{a}$$

which contradicts the irrationality of  $\sqrt{2}$ . So, once again,  $a = 0$  and we see at once that  $c = 0$  as well.

- c. Prove that if  $m, n, p$  and  $q$  are integers then it is impossible to have

$$\frac{\sqrt{2} - m}{n} = \frac{\sqrt{3} - p}{q}$$

and deduce that if  $\alpha$  is any real number and if  $H = \{n\alpha \mid n \in \mathbf{Z}\}$  then the subgroup  $H + Z$  cannot contain both of the numbers  $\sqrt{2}$  and  $\sqrt{3}$ .

Solution: The equation

$$\frac{\sqrt{2} - m}{n} = \frac{\sqrt{3} - p}{q}$$

implies that

$$q\sqrt{2} = n\sqrt{3} - np + mq$$

which, by part b, tells us that

$$0 = q = n = mq - np$$

which is clearly impossible since  $n$  and  $q$  appear denominators of the fractions in the equation

$$\frac{\sqrt{2} - m}{n} = \frac{\sqrt{3} - p}{q}.$$

Now, to obtain a contradiction, suppose that the subgroup  $H + Z$  contains both of the numbers  $\sqrt{2}$  and  $\sqrt{3}$ . Choose integers  $m$  and  $n$  such that

$$\sqrt{2} = m + n\alpha$$

and choose integers  $p$  and  $q$  such that

$$\sqrt{3} = p + q\alpha.$$

Since  $\sqrt{2}$  is irrational, we know that  $\sqrt{2} \neq m$  and so  $n \neq 0$ ; and we know similarly that  $q \neq 0$ . Thus

$$\frac{\sqrt{2} - m}{n} = \alpha = \frac{\sqrt{3} - p}{q}$$

which we know to be impossible.

- d. Suppose that  $G$  is a subgroup of  $R$  other than  $\{0\}$ , that

$$p = \inf\{x \in G \mid x > 0\}$$

and that the number  $p$  is positive. Prove that the set  $G$  is closed.

Solution: We know from an earlier exercise that

$$G = \{np \mid n \in \mathbf{Z}\}.$$

- e. Prove that if  $G$  is a subgroup of  $R$  other than  $\{0\}$  and that  $G$  has no least positive member then  $\overline{G} = R$ .

Solution: This fact was established in an earlier exercise.

- f. Suppose that  $\alpha$  is an irrational number, that

$$H = \{n\alpha \mid n \in \mathbf{Z}\}$$

and that  $G = H + \mathbf{Z}$  (in the sense of this exercise). Prove that although the sets  $H$  and  $\mathbf{Z}$  are closed subgroups of  $R$  and although the set  $G$  is also a subgroup of  $R$ , the set  $G$  is not closed.

Solution: Since  $G$  cannot contain both of the numbers  $\sqrt{2}$  and  $\sqrt{3}$  we know that  $G \neq R$ . To show that  $G$  is not closed we shall make the observation that  $\overline{G} = R$  and, for this purpose, all we have to show is that if

$$p = \inf\{x \in G \mid x > 0\}$$

then  $p = 0$ . Suppose that  $p$  is defined in this way and, to obtain a contradiction, suppose that  $p > 0$ . We know that

$$G = \{np \mid n \in \mathbf{Z}\}$$

and, using the fact that both of the numbers 1 and  $\alpha$  belong to  $G$ , we choose integers  $m$  and  $n$  such that

$$1 = mp$$

and

$$\alpha = np.$$

From the fact that  $p = 1/m$  we see that  $p$  is rational but from the fact that  $p = \alpha/n$  we see that  $p$  must be irrational. Thus we have arrived at the promised contradiction.