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1. Suppose that
$$S = [0,1) \cup (1,2).$$

   a. What is the set of interior points of S?
   The set of interior points of S is $(0,1) \cup (1,2)$.

   b. Given that U is the set of interior points of S, evaluate $\overline{U}$.
   $$\overline{(0,1) \cup (1,2)} = [0,1] \cup [1,2] = \overline{S}.$$
   The purpose of parts a and b is to exhibit a set S such that, if U is the set of interior points of S then $\overline{U} = \overline{S}$.

   c. Give an example of a set S of real numbers such that if U is the set of interior points of S then $\overline{U} \neq \overline{S}$.
   We could take S to be a singleton like $\{3\}$ or it could be the set of all integers. It could also be the set of all rational numbers between 0 and 1.

   d. Give an example of a subset S of the interval $[0,1]$ such that $\overline{S} = [0,1]$ but if U is the set of interior points of S then $\overline{U} \neq [0,1]$.
   Once again, take the set of all rational numbers between 0 and 1.

2. Given that
$$S = \left\{\frac{1}{n} \mid n \in \mathbf{Z}^+\right\},$$
evaluate $\overline{S}$.
Hint: Show that
$$\overline{S} = \{0\} \cup \left\{\frac{1}{n} \mid n \in \mathbf{Z}^+\right\}.$$
First show that $0 \in \overline{S}$. Then observe that every negative number belongs to the set $(-\infty, 0)$ and that if x is any positive number then x belongs to the interval
$$\left(\frac{1}{n+1}, \frac{1}{n}\right)$$
for some positive integer n.

3. Given that S is a set of real numbers, that H is a closed set and that $S \subseteq H$, prove that $\overline{S} \subseteq H$.
We could argue that $\overline{S} \subseteq \overline{H}$ and that, because H is closed, $\overline{H} = H$.

4. Given two sets A and B of real numbers, prove that
$$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$
Solution: Since $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$ we have
$$A \cup B \subseteq \overline{A} \cup \overline{B}.$$
and therefore, since the union of the two closed sets $\overline{A}$ and $\overline{B}$ is closed we have
$$\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}.$$
On the other hand, since A is included in the closed set $\overline{A \cup B}$ we have
$$\overline{A} \subseteq \overline{A \cup B}$$
and, similarly we can see that
$$\overline{B} \subseteq \overline{A \cup B}$$
and so
$$\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}.$$
Therefore
$$\overline{A \cup B} = \overline{A} \cup \overline{B}.$$

5. Given two sets A and B of real numbers, prove that

$$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}.$$

Do the two sides of this inclusion have to be equal? What if A and B are open? What if they are closed?
Since $A \cap B \subseteq A$ we have $\overline{A \cap B} \subseteq \overline{A}$ and similarly that $\overline{A \cap B} \subseteq \overline{B}$. Thus

$$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}.$$

Now observe that if $A = (0,1)$ and $B = (1,2)$ then

$$\overline{A} \cap \overline{B} = [0,1] \cap [1,2] = \{1\}$$

and

$$\overline{A \cap B} = \overline{\emptyset} = \emptyset.$$

Of course, we could give more spectacular examples like $A = Q$ and $B = R \setminus Q$.

6. Prove that if S is any set of real numbers then the set $R \setminus \overline{S}$ is the set of interior points of the set $R \setminus S$.
Most students should be encouraged to write two separate arguments here. The first task is to show that every member of the set $R \setminus \overline{S}$ must be an interior point of $R \setminus S$. Then one should show that every interior point of $R \setminus S$ must belong to $R \setminus \overline{S}$.
On the other hand, a strong student could be permitted to observe that if x is any given number then the statement that x does not belong to $\overline{S}$ is the statement that there exists a number $\delta > 0$ such that $(x-\delta, x+\delta) \cap S = \emptyset$, and that the latter equation is just the condition that $(x-\delta, x+\delta) \subseteq R \setminus S$.

7. Given that α is an upper bound of a given set S of real numbers, prove that the following two conditions are equivalent:

   a. We have $\alpha = \sup S$.

   b. We have $\alpha \in \overline{S}$.

   To prove that condition a implies condition b we assume that $\alpha = \sup S$. We need to show that $\alpha \in \overline{S}$. Suppose that $\delta > 0$. Using the fact that α is the least upper bound of S and that $\alpha - \delta < \alpha$ we choose a member x of S such that $\alpha - \delta < x$. Since $x \in (\alpha-\delta, \alpha+\delta) \cap S$ we have $(\alpha-\delta, \alpha+\delta) \cap S \neq \emptyset$.
   To prove that condition b implies condition a we assume that $\alpha \in \overline{S}$. We need to show that α is the least upper bound of S. Suppose that $p < \alpha$. Since the set (p, ∞) is a neighborhood of α we have $(p,\infty) \cap S \neq \emptyset$. Thus, since α is an upper bound of S and since no number $p < \alpha$ can be an upper bound of S we conclude that α is the least upper bound of S.

8. Is it true that if A and B are sets of real numbers and

$$\overline{A} = \overline{B} = \mathbf{R}$$

then $\overline{A \cap B} = R$?
The answer is no. Look at $A = Q$ and $B = R \setminus Q$.

9. Prove that if A and B are open sets and

$$\overline{A} = \overline{B} = \mathbf{R}$$

then $\overline{A \cap B} = R$. What if only one of the sets A and B is open?
Solution: All we need to know is that at least one of the sets A and B is open. Suppose that A and B are sets of real numbers, that

$$\overline{A} = \overline{B} = \mathbf{R}$$

and that the set A is open.
To prove that

$$\overline{A \cap B} = \mathbf{R},$$

suppose that x is any real number and that $\delta > 0$. Since $x \in \overline{A}$ we know that the set

$$(x-\delta, x+\delta) \cap A$$

is nonempty and we also know that this set is open. Therefore, since $\overline{B} = R$ we know that

$$(x-\delta, x+\delta) \cap A \cap B \neq \emptyset.$$

We have therefore shown that every real number must belong to $\overline{A \cap B}$.

10. Two sets A and B are said to be separated from each other if

$$\overline{A} \cap B = A \cap \overline{B} = \emptyset.$$

Which of the following pairs of sets are separated from each other?

a. $[0,1]$ and $[2,3]$. Yes.

b. $(0,1)$ and $(1,2)$. Yes.

c. $(0,1]$ and $(1,2)$. No because $(0,1] \cap \overline{(1,2)} = \{1\} \neq \emptyset$.

d. Q and $R \setminus Q$. No.

11. Prove that if A and B are closed and disjoint from one another then A and B are separated from each other.
Suppose that A and B are closed and disjoint from one another. Since $A = \overline{A}$ and $B = \overline{B}$, the fact that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ follows at once from the fact that $A \cap B = \emptyset$.

12. Prove that if A and B are open and disjoint from one another then A and B are separated from each other.
Suppose that A and B are open and disjoint from one another. Given any number $x \in A$, we deduce from the fact that A is a neighborhood of x and $A \cap B = \emptyset$ that x is not close to B. Therefore $A \cap \overline{B} = \emptyset$ and we see similarly that $\overline{A} \cap B = \emptyset$.

13. Suppose that S is a set of real numbers. Prove that the two sets S and $R \setminus S$ will be separated from each other if and only if the set S is both open and closed. What then do we know about the sets S for which S and $R \setminus S$ are separated from each other?
Suppose that S and $R \setminus S$ are separated from each other. To show that S is open, suppose that $x \in S$. Since $S \cap \left(\overline{\mathbf{R} \setminus S}\right) = \emptyset$ we know that x is not close to $R \setminus S$. Choose $\delta > 0$ such that

$$(x-\delta, x+\delta) \cap (\mathbf{R} \setminus S) = \emptyset$$

and observe that $(x-\delta, x+\delta) \subseteq S$. Thus S is open and a similar argument shows that $R \setminus S$ is also open. We therefore know that if the sets S and $R \setminus S$ are separated from one another then S is both open and closed.
Now suppose that S is both open and closed. Since the two set S and $R \setminus S$ are closed and disjoint from one other they are separated from one another.

14. This exercise refers to the notion of a subgroup of R that was introduced in an earlier exercise. That exercise should be completed before you start this one.

a. Given that H and K are subgroups of R, prove that the set $H+K$ defined in the sense of an earlier exercise is also a subgroup of R.
To prove that $H+K$ is a subgroup of R we need to show that $H+K$ is nonempty and that the sum and difference of any members of $H+K$ must always belong to $H+K$.
To show that $H+K$ is nonempty we use the fact that H and K are nonempty to choose $x \in H$ and $y \in K$. Since $x+y \in H+K$ we have $H+K \neq \emptyset$.
Now suppose that w_1 and w_2 are any members of the set $H+K$. Choose members x_1 and x_2 of H and members y_1 and y_2 of K such that $w_1 = x_1 + y_1$ and $w_2 = x_2 + y_2$. Since the numbers $x_1 + x_1$ and $x_1 - x_2$ belong to H and the numbers $y_1 + y_2$ and $y_1 - y_2$ belong to K, and since

$$w_1 + w_2 = (x_1 + x_2) + (y_1 + y_2)$$

and

$$w_1 - w_2 = (x_1 - x_2) + (y_1 - y_2)$$

we see at once that $w_1 + w_2$ and $w_1 - w_2$ belong to $H+K$.

b. Prove that if a, b and c are integers and if

$$a\sqrt{2} = b\sqrt{3} + c$$

then $a = b = c = 0$.
Solution: From the equation

$$a\sqrt{2} = b\sqrt{3} + c$$

we see that

$$2a^2 = 3b^2 + 2bc\sqrt{3} + c^2.$$

Therefore, unless $bc = 0$ we have

$$\sqrt{3} = \frac{2a^2 - 3b^2 - c^2}{2bc}$$

which contradicts the fact that the number $\sqrt{3}$ is irrational. Therefore at least one of the number b and c must be zero.

In the event that $c = 0$, the equation

$$a\sqrt{2} = b\sqrt{3} + c$$

becomes

$$a\sqrt{2} = b\sqrt{3}$$

and, unless $a = 0$, the latter equation gives us

$$\sqrt{\frac{2}{3}} = \frac{b}{a}$$

which contradicts the fact that $\sqrt{\frac{2}{3}}$ is irrational. So in the case $c = 0$ we also have $a = 0$ and we see at once that $b = 0$ as well.

In the event that $b = 0$, the equation

$$a\sqrt{2} = b\sqrt{3} + c$$

becomes

$$a\sqrt{2} = c$$

and, unless $a = 0$, the latter equation gives us

$$\sqrt{2} = \frac{c}{a}$$

which contradicts the irrationality of $\sqrt{2}$. So, once again, $a = 0$ and we see at once that $c = 0$ as well.

c. Prove that if m, n, p and q are integers then it is impossible to have

$$\frac{\sqrt{2} - m}{n} = \frac{\sqrt{3} - p}{q}$$

and deduce that if α is any real number and if $H = \{n\alpha \mid n \in \mathbf{Z}\}$ then the subgroup $H + Z$ cannot contain both of the numbers $\sqrt{2}$ and $\sqrt{3}$.

Solution: The equation

$$\frac{\sqrt{2} - m}{n} = \frac{\sqrt{3} - p}{q}$$

implies that

$$q\sqrt{2} = n\sqrt{3} - np + mq$$

which, by part b, tells us that

$$0 = q = n = mq - np$$

which is clearly impossible since n and q appear denominators of the fractions in the equation

$$\frac{\sqrt{2} - m}{n} = \frac{\sqrt{3} - p}{q}.$$

Now, to obtain a contradiction, suppose that the subgroup $H + Z$ contains both of the numbers $\sqrt{2}$ and $\sqrt{3}$. Choose integers m and n such that

$$\sqrt{2} = m + n\alpha$$

and choose integers p and q such that

$$\sqrt{3} = p + q\alpha.$$

Since $\sqrt{2}$ is irrational, we know that $\sqrt{2} \neq m$ and so $n \neq 0$; and we know similarly that $q \neq 0$. Thus

$$\frac{\sqrt{2} - m}{n} = \alpha = \frac{\sqrt{3} - p}{q}$$

which we know to be impossible.

d. Suppose that G is a subgroup of R other than $\{0\}$, that

$$p = \inf\{x \in G \mid x > 0\}$$

and that the number p is positive. Prove that the set G is closed.
Solution: We know from an earlier exercise that

$$G = \{np \mid n \in \mathbf{Z}\}.$$

e. Prove that if G is a subgroup of R other than $\{0\}$ and that G has no least positive member then $\overline{G} = R$.
Solution: This fact was established in an earlier exercise.

f. Suppose that α is an irrational number, that

$$H = \{n\alpha \mid n \in \mathbf{Z}\}$$

and that $G = H + Z$ (in the sense of this exercise). Prove that although the sets H and Z are closed subgroups of R and although the set G is also a subgroup of R, the set G is not closed.
Solution: Since G cannot contain both of the numbers $\sqrt{2}$ and $\sqrt{3}$ we know that $G \neq R$. To show that G is not closed we shall make the observation that $\overline{G} = R$ and, for this purpose, all we have to show is that if

$$p = \inf\{x \in G \mid x > 0\}$$

then $p = 0$. Suppose that p is defined in this way and, to obtain a contradiction, suppose that $p > 0$. We know that

$$G = \{np \mid n \in \mathbf{Z}\}$$

and, using the fact that both of the numbers 1 and α belong to G, we choose integers m and n such that

$$1 = mp$$

and

$$\alpha = np.$$

From the fact that $p = 1/m$ we see that p is rational but from the fact that $p = \alpha/n$ we see that p must be irrational. Thus we have arrived at the promised contradiction.