

About Recursive Sequences

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A Question

Let $x_{n+1} = x_n + \sin x_n$ and $x_0 =$ any number. Then discuss if the sequence $\{x_n\}$ converges or diverges.

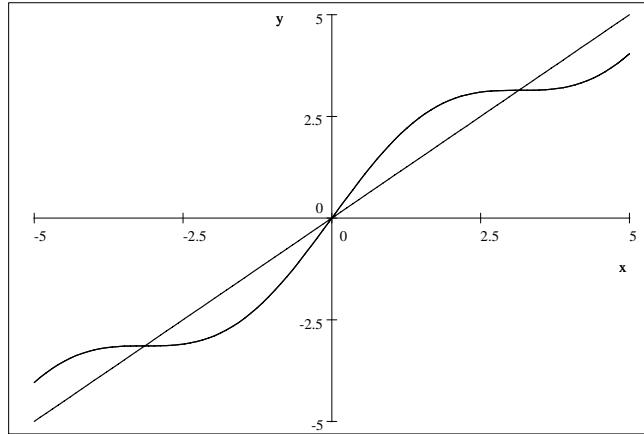
Exploration

We define $f(x) = x + \sin x$ and we find the following iterations by using different initial values:

	1	0	4	5	6
	1. 8415	0	3. 2432	4. 0411	5. 7206
	2. 8051	0	3. 1418	3. 2581	5. 1872
	3. 1353	0	3. 1416	3. 1419	4. 2978
	3. 1416	0	3. 1416	3. 1416	3. 3825
Iterates:	3. 1416	Iterates: 0	Iterates: 3. 1416	Iterates: 3. 1416	Iterates: 3. 1439
	3. 1416	0	3. 1416	3. 1416	3. 1416
	3. 1416	0	3. 1416	3. 1416	3. 1416
	3. 1416	0	3. 1416	3. 1416	3. 1416
	3. 1416	0	3. 1416	3. 1416	3. 1416
	3. 1416	0	3. 1416	3. 1416	3. 1416

6.2			
6.1169			
5.9514			
5.6257			
5.0145			
Iterates: 4.0598			
3.2653			
3.1419			
3.1416			
3.1416			
3.1416			
6.28			
6.2768			
6.2704			
6.2577			
6.2322			
6.1813	10	-6	-20
6.0796	9.4560	-5.7206	-20.913
5.8773	9.4248	-5.1872	-21.794
5.4825	9.4248	-4.2978	-21.990
4.7647	9.4248	-3.3825	-21.991
Iterates: 3.7661	Iterates: 9.4248	Iterates: -3.1439	Iterates: -21.991
3.1814	9.4248	-3.1416	-21.991
3.1416	9.4248	-3.1416	-21.991
3.1416	9.4248	-3.1416	-21.991
3.1416	9.4248	-3.1416	-21.991
3.1416	9.4248	-3.1416	-21.991
3.1416			
3.1416			
3.1416			
3.1416			
3.1416			

The graphs of $y = x$ and $y = f(x)$ are shown below.



Observations

It is interesting to see that the fixed point iteration does not converge to $x = 2k\pi$, no matter how we pick the initial condition. Why?? After all, .

1. Does the limit of $\{x_n\}$ exist?
2. If it does, what is the limit?

Conjectures:

1. The sequence $\{x_n\}$ will converge to either 0 , $(2k + 1)\pi$, or $-(2k + 1)\pi$, where $k = 0, 1, \dots$. It depends on the initial value.

Remarks:

1. *How come the sequence $\{x_n\}$ does not converge to $2k\pi$ (where k is a positive or negative integer) even though the intersections of $y = x$ and $y = f(x) = x + \sin x$ show the intersections at every $k\pi$.*
2. How do we formulate a proper theorem?
3. Show theoretically that $\{x_n\}$ converges.

Theorems

1. If $x_{n+1} = f(x_n)$ converges, where f is a continuous function, then the limit will be one of the solutions of $f(x) = x$.
2. Let $f : [a, b] \rightarrow [a, b]$ be continuous, then the function f must have at least one fixed point of $[a, b]$.
3. Let $f : [a, b] \rightarrow [a, b]$ be continuous and $\delta < 1$. If $|f'(x)| \leq \delta$. Then the function f can have at most one fixed point in $[a, b]$.
4. If $x_{n+1} = f(x_n)$ converges, and f satisfies conditions of '3' above, then $\lim_{n \rightarrow \infty} x_n$ is the solution of $f(x) = x$.

Remark: The theorems above can be generalized to higher dimensions. We will omit the statements here but we refer readers to the section of 'two dimensional examples' below.

More one dimensional examples

Exercise 1. Discuss the convergence or divergence of the sequence $x_{n+1} = f(x_n)$, where $f(x) = ux(1-x)$, and u satisfies one of the following three cases.

Case 1. If $u < 3$:

Case 2. If $u = 3$

Case 3. If $u > 3$.

Exercise 2. Discuss the convergence or divergence of the sequence $x_{n+1} = f(x_n)$, where $f(x) = cx^2$, and c satisfies one of the following three cases.

Case 1. If $c > \frac{1}{4}$:

Case 2. If $c = \frac{1}{4}$

Case 3. If $c < \frac{1}{4}$.

Exercise 3. Consider the sequence defined by the function $f(x) = \arctan x$.

Two dimensional examples

1. Let $f(x, y) = (1 + \frac{1}{3} \cos xy, 1 + \frac{1}{3} \cos(x + y))$. Define

$$x_{n+1} = 1 + \frac{1}{3} \cos x_n y_n,$$

$$y_{n+1} = 1 + \frac{1}{3} \cos(x_n + y_n),$$

$$(x_0, y_0) = (0, 0).$$

Then prove that $f(x, y) = (x, y)$ has a unique solution. Find $\lim_{n \rightarrow \infty} (x_n, y_n)$.

A Contraction Function of Two Variables

$$f(x, y) = \left(1 + \frac{1}{3} \cos xy, 1 + \frac{1}{3} \cos(x + y) \right)$$

for $0 \leq x \leq 2$ and $0 \leq y \leq 2$. We write the coordinates of $f(x, y)$ as $f_1(x, y)$ and $f_2(x, y)$ for each point (x, y) in the square $[0, 2] \times [0, 2]$. We observe that

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} -\frac{y}{3} \sin xy & -\frac{x}{3} \sin xy \\ -\frac{1}{3} \sin(x + y) & -\frac{1}{3} \sin(x + y) \end{bmatrix}.$$

Now point at the expression

$$\left\| \begin{bmatrix} -\frac{y}{3} \sin xy & -\frac{x}{3} \sin xy \\ -\frac{1}{3} \sin(x + y) & -\frac{1}{3} \sin(x + y) \end{bmatrix} \right\|$$

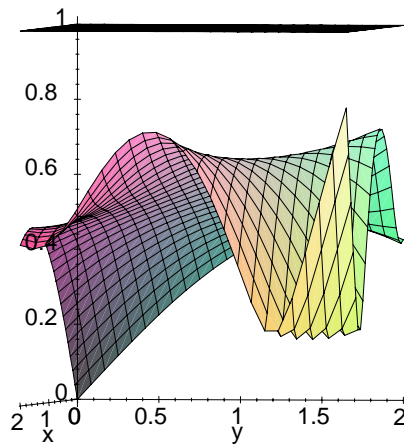
and click on **Evaluate**. The norm is shown as the larger of two identical expressions and is therefore equal to either one of them. To obtain an upper bound for the norm of the matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

as the point (x, y) varies through the square $[0, 2] \times [0, 2]$ we need to maximize the function g defined by the equation

$$g(x, y) = \frac{1}{6} \sqrt{2(\sin^2 xy)y^2 + 2(\sin^2 xy)x^2 + 4 \sin^2(x + y) + 2\sqrt{((\sin^4 xy)y^4 + 2(\sin^4 xy)y^2x^2 + (\sin^4 xy)x^4 + 4 \sin^4(x + y))}}$$

Point at this equation and click on **Define** and **New Definition** and then click on **Plot 3D** and **Rectangular**. Open the plot properties dialog box and set the x and y intervals to be $[0, 2]$. Then drag the expression 1 into the sketch in order to add the plot $z = 1$ to what we already have. After you have rotated the plot suitably it will appear as follows,



and we see from this figure that the maximum value of the function g is less than 1. We can therefore deduce from an earlier subsection that the function f is a contraction.

To show that the fixed point of this function is approximately $(1.1789, .852)$, point at the system of equations

$$1 + \frac{1}{3} \cos xy = x$$

$$1 + \frac{1}{3} \cos(x + y) = y$$

and click on **Solve** and **Numeric**. Finally, we shall approximate this fixed point by iterating the function f starting at an arbitrary point in the square $[0, 2] \times [0, 2]$. Unfortunately, *Scientific Notebook* does not provide automatic iteration of a function of two variables but we can still iterate the function by hand. We see that

$$\begin{aligned}
f(0,0) &= (1.3333, 1.3333) \\
f(1.3333, 1.3333) &= (.93153, .70357) \\
f(.93153, .70357) &= (1.2643, .97858) \\
f(1.2643, .97858) &= (1.1091, .79246) \\
f(1.1091, .79246) &= (1.2127, .89174) \\
f(1.1091, .79246) &= (1.2127, .89174) \\
f(1.2127, .89174) &= (1.1567, .83044) \\
f(1.1567, .83044) &= (1.191, .86519) \\
f(1.191, .86519) &= (1.1715, .84448) \\
f(1.1715, .84448) &= (1.1831, .85646) \\
f(1.1831, .85646) &= (1.1764, .84941) \\
f(1.1764, .84941) &= (1.1803, .85351) \\
f(1.1803, .85351) &= (1.178, .85112) \\
f(1.178, .85112) &= (1.1794, .85252) \\
f(1.1794, .85252) &= (1.1786, .85168) \\
f(1.1786, .85168) &= (1.179, .85217) \\
f(1.179, .85217) &= (1.1788, .85191) \\
f(1.1788, .85191) &= (1.1789, .85204) \\
f(1.1789, .85204) &= (1.1788, .85197) \\
f(1.1788, .85197) &= (1.1789, .85203) \\
f(1.1789, .85203) &= (1.1788, .85198)
\end{aligned}$$

and this array of iterates gives us about the same approximation to the fixed point.