Fixed Point Theorem and Sequences in One or Two Dimensions

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Let us consider a recursive sequence of $x_{n+1} = x_n + \sin x_n$ and the initial value x_0 can be any real number. Then we would like to ask if the sequence $\{x_n\}$ converges or not. We explore the following possibilities by starting with various initial values of x_0. The first number of each column below is the initial value for $x_{n+1} = x_n + \sin x_n$.

1 Exploration

We observe the following iterations by using different initial values: For example, if we start with $x_0 = 1$, the table suggests the sequence converges to π. If $x_0 = 10$, we note that the sequence seems to converge to 9.4248, which is close to 3π.

$$\begin{bmatrix} 1 \\ 1.8415 \\ 2.8051 \\ 3.1353 \\ 3.1416 \\ 3.1416 \\ 3.1416 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 4 \\ 3.2432 \\ 3.1418 \\ 3.1416 \\ 3.1416 \end{bmatrix} \begin{bmatrix} 5 \\ 4.0411 \\ 3.2581 \\ 3.1419 \\ 3.1416 \end{bmatrix} \begin{bmatrix} 6 \\ 5.7206 \\ 5.1872 \\ 4.2978 \\ 3.3825 \\ 3.1439 \\ 3.1416 \\ 3.1416 \end{bmatrix} \begin{bmatrix} 6.2 \\ 6.1169 \\ 5.9514 \\ 5.6257 \\ 5.0145 \\ 4.0598 \\ 3.2653 \\ 3.1419 \\ 3.1416 \end{bmatrix} \begin{bmatrix} 10 \\ 9.4560 \\ 9.4248 \\ 9.4248 \\ 9.4248 \end{bmatrix} \begin{bmatrix} -6 \\ -5.7206 \\ -5.1872 \\ -4.2978 \\ -3.3825 \\ -3.1439 \\ -3.1416 \\ -3.1416 \end{bmatrix}$$

$$\begin{bmatrix} 6.28 \\ 6.2768 \\ 6.2704 \\ 6.2577 \\ 6.2322 \\ 6.1813 \\ 6.0796 \\ 5.8773 \\ 5.4825 \\ 4.7647 \\ 3.7661 \\ 3.1814 \\ 3.1416 \\ 3.1416 \end{bmatrix}$$

It is interesting to see if we start with $x_0 = 6.28$, the sequence seems to converge to π.Next, did you notice that if an iteration suggests a convergence, the limit would have to lie at the intersection of $y = x$ and $y = x + \sin x$, which we sketch both graphs below:

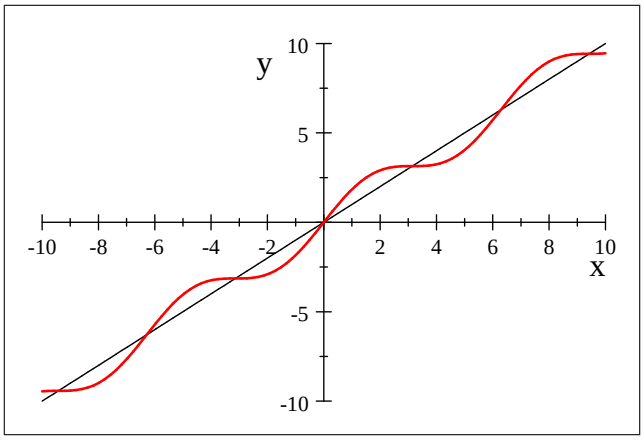


Figure 1. Graph of $y = x$ and $y = x + \sin x$.

1.1 Observations

The sequence $\{x_n\}$ will converge to either 0, $(2k+1)\pi$, or $-(2k+1)\pi$, where $k = 0, 1, ..$, depending on the initial values for the sequence.

Remarks:

1. Why does the sequence $\{x_n\}$ not converge to $2k\pi$ (where k is a positive or negative integer) even though the intersections of $y = x$ and $y = f(x) = x + \sin x$ show the intersections at every $k\pi$.

2. How do we formulate a proper theorem relating a sequence and a function?

3. Can you recall how we show theoretically a sequence $\{x_n\}$ converges?

1.2 Definition of a Fixed Point

If f is a function from a set A into a set B then a point a in the set A is said to be a fixed point of the function f if $f(a) = a$.

Thus, if A is a set of real numbers then a number a is a fixed point of f when the graph $y = f(x)$ meets the line $y = x$ at $x = a$.

1.3 Fixed Point Theorem

Suppose that $f : [a, b] \to [a, b]$ is a continuous and satisfying $|f'(x)| \leq \delta < 1$ for each $x \in [a, b]$. Then the function f can have at most one fixed point in $[a, b]$.

Example 1 *We consider the function f defined by*

$$f(x) = \pi + \frac{\cos x}{2}$$

for every $x \in [0, 2\pi]$. *Observe that* $f : [0, 2\pi] \to [0, 2\pi]$. *Now since*

$$|f'(x)| = \left| \frac{\sin x}{2} \right| \leq \frac{1}{2}$$

for each x *we know that* f *is a contraction on the interval* $[0, 2\pi]$ *and therefore* f *must have a unique fixed point in this interval.*

1.4 Newton's Method

Let f be a differentiable function on (a, b).Choose a point x_1 near a root r of f. Define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

If $\lim_{n\to\infty} x_n$ exists, then $\lim_{n\to\infty} x_n = r$.

Example 2 *Exercises for Newton's method:*

http://archives.math.utk.edu/visual.calculus/3/newton.4/index.html

Problem 3 *How do you derive the Newton's algorithm?*

Problem 4 *Describe how you can solve a fixed point problem by using the Newton's Method.*

Problem 5 *Describe how you can turn a Newton's Method into a Fixed Point problem.*

1.5 Theorems

1. If $x_{n+1} = f(x_n)$ converges, where $f : [a, b] \to [a, b]$ is a continuous function, then the limit will be one of the solutions of $f(x) = x$.

2. Let $f : [a, b] \to [a, b]$ be continuous, then the function f must have at least one fixed point of $[a, b]$.

3. Let $f : [a, b] \to [a, b]$ be continuous and $\delta < 1$. If $|f'(x)| \leq \delta$. Then the function f can have at most one fixed point in $[a, b]$.

4. If $x_{n+1} = f(x_n)$ converges, and f satisfies conditions 3 above, then $\lim_{n\to\infty} x_n$ is the solution of $f(x) = x$.

Remark: The theorems above can be generalized to higher dimensions. We will omit the statements here but we refer readers to the section of 'two dimensional examples' below.

1.6 More one dimensional examples

Exercise 1. Discuss the convergence or divergence of the sequence $x_{n+1} = f(x_n)$, where $f(x) = ux(1-x)$, and u satisfies one of the following three cases.

Case 1. If $u < 3$:

Case 2. If $u = 3$

Case 3. If $u > 3$.

Exercise 2. Discuss the convergence or divergence of the sequence $x_{n+1} = f(x_n)$, where $f(x) = cx^2$, and c satisfies one of the following three cases.

Case 1. If $c > \frac{1}{4}$:

Case 2. If $c = \frac{1}{4}$

Case 3. If $c < \frac{1}{4}$.

Exercise 3. Consider the sequence defined by the function $f(x) = \arctan x$.

2 Two dimensional examples

1. Let $f(x,y) = (1 + \frac{1}{3}\cos xy, 1 + \frac{1}{3}\cos(x+y))$. Define

$$\begin{aligned} x_{n+1} &= 1 + \frac{1}{3}\cos x_n y_n, \\ y_{n+1} &= 1 + \frac{1}{3}\cos(x_n + y_n), \\ (x_0, y_0) &= (0,0). \end{aligned}$$

Then prove that $f(x,y) = (x,y)$ has a unique solution. Find $\lim_{n\to\infty}(x_n, y_n)$.

2.0.1 A Contraction Function of Two Variables

$$f(x,y) = \left(1 + \frac{1}{3}\cos xy, 1 + \frac{1}{3}\cos(x+y)\right)$$

for $0 \le x \le 2$ and $0 \le y \le 2$. We write the coordinates of $f(x,y)$ as $f_1(x,y)$ and $f_2(x,y)$ for each point (x,y) in the square $[0,2] \times [0,2]$. We observe that

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} -\frac{y}{3}\sin xy & -\frac{x}{3}\sin xy \\ -\frac{1}{3}\sin(x+y) & -\frac{1}{3}\sin(x+y) \end{bmatrix}.$$

Now point at the expression

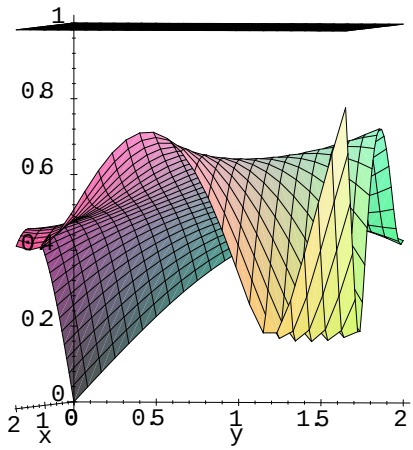
$$\left\| \begin{bmatrix} -\frac{y}{3}\sin xy & -\frac{x}{3}\sin xy \\ -\frac{1}{3}\sin(x+y) & -\frac{1}{3}\sin(x+y) \end{bmatrix} \right\|$$

and click on Evaluate. The norm is shown as the larger of two identical expressions and is therefore equal to either one of them. To obtain an upper bound for the norm of the matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

as the point (x,y) varies through the square $[0,2] \times [0,2]$ we need to maximize the function g defined by the equation

$g(x,y) = \frac{1}{6}\sqrt{\left(\begin{array}{c} 2\left(\sin^2 xy\right)y^2 + 2\left(\sin^2 xy\right)x^2 + 4\sin^2(x+y) + \\ 2\sqrt{\left(\begin{array}{c}\left(\sin^4 xy\right)y^4 + 2\left(\sin^4 xy\right)y^2x^2 + \\ \left(\sin^4 xy\right)x^4 + 4\sin^4(x+y) + 8\left(\sin^2 xy\right)y\left(\sin^2(x+y)\right)x\end{array}\right)}\end{array}\right)}$. Point at this equation and click on Define and New Definition and then click on Plot 3D and Rectangular. Open the plot properties dialog box and set the x and y intervals to be $[0,2]$. Then drag the expression 1 into the sketch in order to add the plot $z = 1$ to what we already have. After you have rotated the plot suitably it will appear as follows,



and we see from this figure that the maximum value of the function g is less than 1. We can therefore deduce from Subsection **??** that the function f is a contraction.

To show that the fixed point of this function is approximately $(1.1789, .852)$, point at the system of equations

$$\begin{aligned} 1 + \frac{1}{3}\cos xy &= x \\ 1 + \frac{1}{3}\cos(x+y) &= y \end{aligned}$$

and click on Solve and Numeric. Finally, we shall approximate this fixed point by iterating the function f starting at an arbitrary point in the square $[0,2] \times [0,2]$. Unfortunately, Scientific Notebook does not provide automatic iteration of a function of two variables but we can still iterate the function by

hand. We see that

$$
\begin{aligned}
f(0,0) &= (1.3333, 1.3333) \\
f(1.3333, 1.3333) &= (.93153, .70357) \\
f(.93153, .70357) &= (1.2643, .97858) \\
f(1.2643, .97858) &= (1.1091, .79246) \\
f(1.1091, .79246) &= (1.2127, .89174) \\
f(1.1091, .79246) &= (1.2127, .89174) \\
f(1.2127, .89174) &= (1.1567, .83044) \\
f(1.1567, .83044) &= (1.191, .86519) \\
f(1.191, .86519) &= (1.1715, .84448) \\
f(1.1715, .84448) &= (1.1831, .85646) \\
f(1.1831, .85646) &= (1.1764, .84941) \\
f(1.1764, .84941) &= (1.1803, .85351) \\
f(1.1803, .85351) &= (1.178, .85112) \\
f(1.178, .85112) &= (1.1794, .85252) \\
f(1.1794, .85252) &= (1.1786, .85168) \\
f(1.1786, .85168) &= (1.179, .85217) \\
f(1.179, .85217) &= (1.1788, .85191) \\
f(1.1788, .85191) &= (1.1789, .85204) \\
f(1.1789, .85204) &= (1.1788, .85197) \\
f(1.1788, .85197) &= (1.1789, .85203) \\
f(1.1789, .85203) &= (1.1788, .85198)
\end{aligned}
$$

and this array of iterates gives us about the same approximation to the fixed point.