Math 431 Test 1 hint.

1. When can we say $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$?

   First, if $f(x) = \sum_{n=1}^{\infty} a_n(x-a)^n$ for all x in an interval containing a, then $a_n = f^{(n)}(a)/n!$.Next, obviously, we need f to be differentiable infinitely many times in an interval containing a,the interval will be the interval of convergence. To find the interval of convergence, we may apply the Ratio Test on the infinite series. In other words, we want to solve for x so that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{f^{(n+1)}(a)(x-a)^{n+1}}{(n+1)!}\cdot\frac{n!}{f^{(n)}(a)(x-a)^n}\right| = \lim_{n\to\infty}\left|\frac{f^{(n+1)}(a)}{f^{(n)}(a)\cdot(n+1)}\cdot(x-a)\right| < 1.$$

   Finally, it follows from the Taylor's Theorem, we need $\lim_{n\to\infty} R_n(x) = 0$ in order for us to write $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

2. Prove that if f is continuous on $[a, b]$, differentiable over (a, b) and if $f'(x) = 0$ for all $x \in (a, b)$, then f is constant over $[a, b]$.

   Proof: First we show that f must be constant in (a, b).Suppose not, there exist $x < y$ such that $f(x) \neq f(y)$.Without loss of generality, we assume $f(x) < f(y)$,it follows from the Mean Value Theorem that there exists a $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x},$$

   which is not 0, a contradiction. Next we want to prove that $f(a) = f(b) =$ constant too. Since f is continuous at $x = a$, we take any sequence $\{x_n\} \cap (a, b) \to a$, we have $f(a) = \lim_{n\to\infty} f(x_n) =$ constant. Similarly, $f(b)$ should be the same constant too.

3. Prove that if $f' < 0$ in (a, b), then f is decreasing in (a, b). [hint: Pick any $x, y \in (a, b)$ and $x < y$.It follows the Mean Value Theorem that there exists a $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x} < 0.$$

   This implies $f(y) - f(x) < 0$ since $y > x$, or f is decreasing in (a, b).]

4. Prove or disprove that if $|f|$ is continuous over an interval (a, b),then so is f. (If it is true, prove it; otherwise, give a counter example).

   Counter example from Cameron: Take $f(x) = 1$ if $x < 0$ and $f(x) = 1$ if $x \geq 0$. Then $|f|$ is continuous at $x = 0$ but f is not.

5. Prove or disprove that if $g \leq f \leq h$ on (a, b), and both g and h are differentiable at a point $c \in (a, b)$,then f is differentiable at $x = c$. [hint: False.]

6. Show that there is some u with $0 < u < 2$ such that $u^2 + \cos(\pi u) = 4$.[hint: Apply the IVT on the function $f(x) = x^2 + \cos(\pi x) - 4$ and note that $f(0)$ and $f(2)$ have different signs or you may apply the Fixed Point Theorem to solve this problem.]

7. Prove that if $f : [a, b] \to [a, b]$ is continuous, then there exists an $x \in [a, b]$ such that $f(x) = x$. [Apply IVT on $h(x) = f(x) - x$ and note that $h(a) \geq 0$ and $h(b) \leq 0$.]

8. If $f(x) = \ln(x)$

   (a) Find the Taylor's expansion for f at $x = 3$ by using degree 4 polynomial.

   (b) Find the interval of convergence of the Taylor series expansion of $f(x)$ at $x = 3$.

   (c) Find the constant c which satisfies the equation from the Taylor's Theorem:

   $$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(c)}{4!}(x-a)^4,$$

   for some c in $[2.5, 3]$.[hint: The interval of convergence is $(0, 6)$ or when $|x - 3| < 1$ before checking the endpoints $x = 0$ or 6; for part (c) you will solve an equation but your answer should fall between 2.5 and 3, which is about 2.89...]

9. A function f is well-defined on an interval containing the point $x = c$ and assume

   $$\lim_{x \to c} \frac{f(x) - f(c)}{(x - c)^2} = A < 0.$$

   Then prove that $f'(c)$ exists and f has a maximum at $x = c$.[Hint: First we need to show that $f'(c) = 0$.Note that $f'(c) = \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c} = \lim_{x \to c} \left(\dfrac{f(x) - f(c)}{(x - c)^2} \cdot (x - c) \right) = \left(\lim_{x \to c} \dfrac{f(x) - f(c)}{(x - c)^2} \right) \cdot (\lim_{x \to c}(x - c)) = A \cdot 0 = 0$.Next, we show that f has a relative maximum at $x = c$. To do this, it follows from $\lim_{x \to c} \frac{f(x) - f(c)}{(x-c)^2} = A < 0$ that given $\epsilon > 0$, there exists a $\delta > 0$ such that

   $$-\epsilon + A < \frac{f(x) - f(c)}{(x - c)^2} < A + \epsilon.$$

   Choose $\epsilon = -A/2 > 0$, we can find $\delta^* > 0$ such that if $x \in (c - \delta^*, c + \delta^*)$ we have $\dfrac{f(x) - f(c)}{(x - c)^2} < A + \epsilon = A/2 < 0$. Since $(x - c)^2 > 0$, we have $f(x) - f(c) < 0$ or $f(x) < f(c)$ when $x \in (c - \delta^*, c + \delta^*)$.Thus f has a relative maximum at $x = c$.]

10. Given the parametric curve C of $[\cos(2t+2), 2\cos(t)]$ with $t \in [0, \pi]$, and the point A is when $t = 0$, the point B is when $t = 1.75$.

   (a) Use the Cauchy Mean Value Theorem to find appropriate value t where the slope of the tangent line at a point P on C is equal to the slope of the secant line connecting AB.

   (b) Find the corresponding parametric curve C' of C where we may apply the Rolle's Theorem by rotating AB to a horizontal line segment.