

Section 3.7

Derivatives of Logarithmic Functions

Review

$$\log_b a = x \Leftrightarrow b^x = a$$

$$\ln x = a \Leftrightarrow e^x = a$$

The derivative of the natural logarithm

Let $y = \ln(x)$

By definition of $\ln(x)$

$$e^y = x$$

$$\frac{d}{dx} e^y = \frac{d}{dx} x$$

$$e^y y' = 1$$

$$\frac{e^y y'}{e^y} = \frac{1}{e^y}$$

$$y' = \frac{1}{x}$$

Definition

$$\frac{d}{dx} (\ln(x)) = \frac{1}{x}$$

Example 1

Find the derivative of $y = 3 \ln x$

$$y = 3 \ln x$$

$$y' = 3 \frac{d}{dx} \ln(x)$$

$$y' = 3 \left(\frac{1}{x} \right)$$

$$y' = \frac{3}{x}$$

Example 2

Find the derivative of $y = x^2 \ln(x)$

Use the product rule

$$y = x^2 \ln(x)$$

$$y' = \frac{d}{dx}(x^2)(\ln(x)) + \frac{d}{dx}\ln(x)(x^2)$$

$$y' = 2x \ln(x) + x^2 \frac{1}{x}$$

$$y' = 2x \ln(x) + x$$

Example 3

Find the derivative of $y = \ln(x^2 - 1)$

$$y = \ln(x^2 - 1)$$

Let $y = \ln u$ where $u = x^2 - 1 \Rightarrow du = 2x$

$$y' = \frac{1}{u} du$$

$$y' = \frac{1}{x^2 - 1} (2x)$$

$$y' = \frac{2x}{x^2 - 1}$$

Example 4

Find the derivative of $y = \ln(3x + 2)$

$$y = \ln(3x + 2) \text{ Let } y = \ln u \text{ where } u = 3x + 2 \Rightarrow du = 3$$

$$y' = \frac{du}{u} = \frac{3}{3x + 2}$$

Example 5

Find the derivative of $y = \ln(\sin(x))$

$$y = \ln(\sin(x))$$

$$y = \ln u$$

$$\text{where } u = \sin x \Rightarrow du = \cos x$$

$$y' = \frac{du}{u} = \frac{\cos(x)}{\sin(x)} = \cot(x)$$

Example 6

Find the derivative of $y = \sqrt[3]{\ln x}$

$$y = \sqrt[3]{\ln x}$$

$$y = (\ln x)^{\frac{1}{3}}$$

$$y = u^{\frac{1}{3}} \text{ where } u = \ln x \Rightarrow du = \frac{1}{x}$$

$$y' = \frac{1}{3} u^{\frac{1}{3}-1} du = \frac{1}{3} u^{-\frac{2}{3}} du = \frac{1}{3} \cdot \frac{1}{u^{\frac{2}{3}}} du = \frac{1}{3} \cdot \frac{1}{\sqrt[3]{u^2}} du = \frac{1}{3} \cdot \frac{1}{\sqrt[3]{(\ln x)^2}} \cdot \frac{1}{x} = \frac{1}{3x\sqrt[3]{(\ln x)^2}}$$

Example 7

Find the derivative of $y = \frac{\ln(2x)}{1 + \ln(2x)}$

$$y = \frac{\ln(2x)}{1 + \ln(2x)}$$

$$y' = \frac{(1 + \ln(2x)) \frac{d}{dx}(\ln(2x)) - \ln(2x) \frac{d}{dx}(1 + \ln(2x))}{(1 + \ln(2x))^2}$$

$$y' = \frac{(1 + \ln(2x)) \left(\frac{2}{2x}\right) - \ln(2x) \left(\frac{2}{2x}\right)}{(1 + \ln(2x))^2}$$

$$y' = \frac{\left(\frac{2}{2x}\right) ((1 + \ln(2x)) - \ln(2x))}{(1 + \ln(2x))^2}$$

$$y' = \frac{2(1 + \ln(2x) - \ln(2x))}{(1 + \ln(2x))^2} = \frac{2}{2x(1 + \ln(2x))^2} = \frac{1}{x(1 + \ln(2x))^2}$$

Definition

$$\frac{d}{dx}(a^x) = a^x \ln a$$

Example 8

Find the derivative of $y = 3^x$

$$y = 3^x$$

$$y' = 3^x \ln(3)$$

Example 9

Find the derivative of $y = 2^x$

$$y = 2^x$$

$$y' = 2^x \ln(2)$$

Find the derivative of $y = \log_a x$

Let $y = \log_a x \Rightarrow a^y = x$

$$a^y = x$$

$$\frac{d}{dx} a^y = \frac{d}{dx} x$$

$$a^y \ln a y' = 1$$

$$y' = \frac{1}{a^y \ln a}$$

$$y' = \frac{1}{x \ln a}$$

Definition

$$\frac{d}{dx} (\log_a x) = \frac{1}{x \ln a}$$

Example 10

Find the derivative of $y = \log_3(x^2 - 1)$

$$y = \log_3(x^2 - 1)$$

$$y = \log_3(u) \text{ where } u = x^2 - 1 \Rightarrow \frac{du}{dx} = 2x$$

$$y' = \frac{1}{u \ln 3} \cdot \frac{du}{dx} = \frac{1}{(x^2 - 1) \ln 3} (2x) = \frac{2x}{(x^2 - 1) \ln 3}$$

Example 11

Find the derivative of $y = \log_6(\sin(2x))$

$$y = \log_6(\sin(2x))$$

$$y = \log_6(u) \text{ where } u = \sin(2x) \Rightarrow \frac{du}{dx} = 2\cos(2x)$$

$$y' = \frac{1}{u \ln 6} \cdot \frac{du}{dx} = \frac{1}{(\sin(2x)) \ln 6} (2\cos(2x)) = \frac{2\cos(2x)}{(\sin(2x)) \ln 6}$$
