Math 151
Section 3.5
Chain Rule

The Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 1

$$y = \left(3x^2 + 5\right)^5$$

$$Let\ y = u^5\ \ where\ u = 3x^2 + 5 \Rightarrow \frac{du}{dx} = 6x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$y' = 5u^4 \frac{du}{dx}$$

$$y' = 5(3x^2 + 5x)^4 (6x)$$

$$y' = (30x)(3x^2 + 5x)^4$$

Example 2

$$y = \left(x^2 + 3x\right)^7$$

$$Let\ y = u^7\ \ where\ u = x^2 + 3x \Rightarrow \frac{du}{dx} = 2x + 3$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$y' = 7u^{7-1} \frac{du}{dx}$$

$$y' = 7u^6 \frac{du}{dx}$$

$$y' = 7(x^2 + 7x)^6 (2x + 3)$$

$$y' = (14x + 21)(x^2 + 3x)^6$$

Example 3

Find the derivative of $y = \sqrt{x^2 - 6x}$

$y = \sqrt{x^2 - 6x}$

$y = (x^2 - 6x)^{\frac{1}{2}}$

Let $y = u^{\frac{1}{2}}$ *where* $u = x^2 - 6x \Rightarrow \frac{du}{dx} = 2x - 6$

$y' = \frac{1}{2}u^{\frac{1}{2}-1}\frac{du}{dx}$

$y' = \frac{1}{2}u^{-\frac{1}{2}}\frac{du}{dx}$

$y' = \frac{1}{2}(x^2 - 6x)^{-\frac{1}{2}}(2x - 6)$

$y' = \frac{2x - 6}{2(x^2 - 6x)^{\frac{1}{2}}}$

$y' = \frac{2x - 6}{\sqrt{x^2 - 6x}}$

Example 4

Find the derivative of $f(x) = \sqrt{x^3 - 3x}$

$y = \sqrt{x^3 - 3x}$

$y = (x^3 - 3x)^{\frac{1}{2}}$

Let $y = u^{\frac{1}{2}}$ *where* $u = x^3 - 3x \Rightarrow \frac{du}{dx} = 3x^2 - 3$

$y = \frac{1}{2}u^{\frac{1}{2}-1}\frac{du}{dx} = \frac{1}{2}u^{-\frac{1}{2}}\frac{du}{dx} = \frac{1}{2}(x^3 - 3x)^{-\frac{1}{2}}(3x^2 - 3) = \frac{3x^2 - 3}{2(x^3 - 3x)^{\frac{1}{2}}} = \frac{3x^2 - 3}{2\sqrt{x^3 - 3x}}$

Example 5

$y = \sqrt[3]{x^2 + 4}$

$y = \left(x^2 + 4\right)^{\frac{1}{3}}$

Let $y = u^{\frac{1}{3}}$ *where* $u = x^2 + 4 \Rightarrow \dfrac{du}{dx} = 2x$

$y' = \dfrac{1}{3}u^{\frac{1}{3}-1}\dfrac{du}{dx}$

$y' = \dfrac{1}{3}u^{-\frac{2}{3}}\dfrac{du}{dx}$

$y' = \dfrac{\dfrac{du}{dx}}{3u^{\frac{2}{3}}}$

$y' = \dfrac{2x}{3(x^2 + 4)^{\frac{2}{3}}}$

$y' = \dfrac{2x}{3\sqrt[3]{x^2 + 4}^2}$

Example 6

Find the derivative of $y = \cos(5x)$

$y = \cos(5x)$

$u = 5x$

$\dfrac{du}{dx} = 5$

$y = \cos u$

$y' = \cos u \cdot \dfrac{du}{dx}$

$y' = 5\cos(5x)$

Example 7

Find the $y = e^{3x^2}$

$y = e^{3x^2}$

$u = 3x^2$

$\frac{du}{dx} = 6x$

$y' = e^u \frac{du}{dx}$

$y' = e^{3x^2}(6x)$

$y' = 6xe^{3x^2}$

Example 8

Find the $y = e^{x^2-6x}$

$y = e^{x^2-6x}$

$u = x^2 - 6x$

$\frac{du}{dx} = 2x - 6$

$y' = e^u \frac{du}{dx}$

$y' = e^{x^2-6x}(2x-6)$

$y' = (2x-6)e^{x^2-6x}$

Example 9

Find the derivative of $y = e^{3x}\cos(6x)$

$y = e^{3x}\cos(6x)$

$$y' = \frac{d}{dx}e^{3x}\cos(6x) + \frac{d}{dx}(\cos(6x))e^{3x}$$
$$y' = 3e^{3x}\cos(6x) - 6e^{3x}\sin(x)$$
$$y' = 3e^{3x}(\cos(6x) - 2\sin(6x))$$

Example 10

Find the derivative of $y = 4xe^{x^2}$

$$y = 4xe^{x^2}$$
$$y' = \frac{d}{dx}(4x)e^{x^2} + \frac{d}{dx}e^{x^2}(4x)$$
$$y' = 4e^{x^2} + (4x)(2x)e^{x^2}$$
$$y = 4e^{x^2} + 8x^2e^{x^2}$$
$$y' = \left(4 + 8x^2\right)e^{x^2}$$

Example 11

Find the derivative of $y = e^{x\sin 4x}$

$$y = e^{x\sin 4x}$$
$$y' = \left(x\sin 4x\right)' e^{x\sin 4x}$$
$$y' = \left(\left(x\right)' \cdot \sin 4x + \left(\sin 4x\right)' \cdot x\right) e^{x\sin 4x}$$
$$y' = \left(\sin 4x + 4\cos 4x \cdot x\right) e^{x\sin 4x}$$
$$y' = e^{x\sin 4x}\sin 4x + 4xe^{xin4x}\cos 4x$$

Example 12

Find the derivative of $y = \tan^2 x$

$$y = \tan^2 x$$
$$y = u^2 \ \ where\ u = \tan x \Rightarrow \frac{du}{dx} = \sec^2 x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$\frac{dy}{dx} = 2u \cdot \frac{du}{dx}$$
$$\frac{dy}{dx} = 2u \cdot \sec^2 x$$
$$\frac{dy}{dx} = 2\tan x \cdot \sec^2 x$$

Example 13

Find the slope of tangent to the function $y = \left(x^2 + 2x\right)^3$ at the point (1,27)

$y = \left(x^2 + 2x\right)^3$

$Let\ y = u^3\ \ where\ u = x^2 + 2x \Rightarrow \frac{du}{dx} = 2x + 2$

$y' = 3u^{3-1}\frac{du}{dx}$

$y' = 3u^2\frac{du}{dx}$

$y' = 3(x^2 + 2x)^2(2x + 2)$

$y' = (6x + 6)(x^2 + 3x)^2$

Slope

$y' = (6(1) + 6)(1^2 + 2(1))^2 = (6 + 6)(1 + 2)^2 = 12(3)^2 = 12(9) = 108$

$m = 108$

Example 14

Find the equation of a tangent line to $y = \cos x + \cos^2 x$ at the point (0,2)

$y = \cos x + (\cos x)^2$

$y' = -\sin x + 2(\cos x)^{2-1}(-\sin x)$

$y' = -\sin x - 2\sin x \cos x$

$y = -\sin(0) - 2\sin(0)\cos(0) = -0 - 2(0)(1) = 0 - 0 = 0$

$y - 2 = 0(x - 0)$

$y - 2 = 0$

$y = 2$