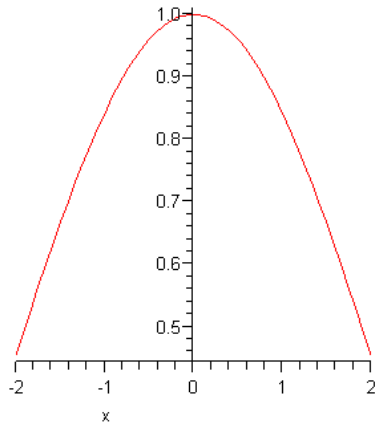


Math 151
Section 3.4

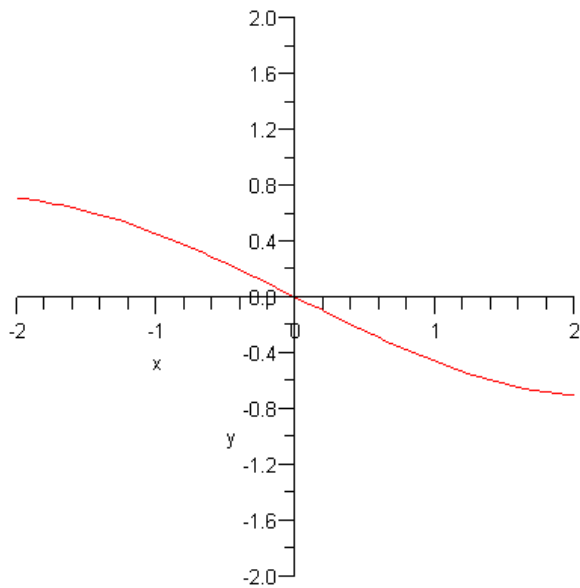
Derivatives of Trigonometric Functions

Review

$$\lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1$$



$$\lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} = 0$$



The Derivative of Sine

$$\begin{aligned}f(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} \\&= \lim_{h \rightarrow 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{\sin(x)\cos(h) - \sin(x)}{h} + \lim_{h \rightarrow 0} \frac{\cos(x)\sin(h)}{h} \\&= \lim_{h \rightarrow 0} \frac{\sin(x)(\cos(h) - 1)}{h} + \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \\&= \sin(x) \lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \\&= \sin(x)(0) + \cos(x)(1) \\&= \cos(x)\end{aligned}$$

Rule 1: If $f(x) = \sin(x)$, $f'(x) = \cos(x)$

The Derivative of Cosine

$$\begin{aligned}f(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} \\&= \lim_{h \rightarrow 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{\cos(x)\cos(h) - \cos(x)}{h} - \lim_{h \rightarrow 0} \frac{\sin(x)\sin(h)}{h} \\&= \lim_{h \rightarrow 0} \frac{\cos(x)(\cos(h) - 1)}{h} - \sin(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \\&= \cos(x) \lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} + \sin(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \\&= \cos(x)(0) - \sin(x)(1) \\&= -\sin(x)\end{aligned}$$

Rule 2: If $f(x) = \cos(x)$, $f'(x) = -\sin(x)$

The Derivative of Tangent

If $f(x) = \tan(x)$, then $f'(x) = \sec^2(x)$

Derivatives of the six trigonometric functions

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

$$\frac{d}{dx}(\tan(x)) = \sec^2(x)$$

$$\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$$

$$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$$

$$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$$

Example 1

Find the derivative of $y = 2\csc(x) + 5\cos(x)$

$$y = 2\csc(x) + 5\cos(x)$$

$$y' = \frac{d}{dx}(2\csc(x)) + \frac{d}{dx}(5\cos(x))$$

$$y' = -2\csc(x)\cot(x) - 5\sin(x)$$

Example 2

Find the derivative of $y = \sqrt{x} \sin(x)$

$$y = \sqrt{x} \sin(x)$$

$$y = x^{\frac{1}{2}} \sin(x)$$

$$y' = \frac{d}{dx} \left(x^{\frac{1}{2}} \right) \sin(x) + \frac{d}{dx} (\sin(x)) \left(x^{\frac{1}{2}} \right)$$

$$y' = \frac{1}{2} x^{\frac{1}{2}-1} \sin(x) + \cos(x) \left(x^{\frac{1}{2}} \right)$$

$$y' = \frac{1}{2} x^{-\frac{1}{2}} \sin(x) + x^{\frac{1}{2}} \cos(x)$$

$$y' = \frac{\sin(x)}{2\sqrt{x}} + \sqrt{x} \cos(x)$$

Example 3

Find the derivative of $y = e^u (\cos u + cu)$

$$y = e^u (\cos u + cu)$$

$$y' = \frac{d}{du} (e^u) (\cos u + cu) + \frac{d}{du} (\cos u + cu) e^u$$

$$y' = e^u (\cos u + cu) + (-\sin u + c) e^u$$

$$y' = e^u \cos u + c u e^u - e^u \sin u - c e^u$$

$$y' = e^u (\cos u - \sin u) - c e^u (u - 1)$$

Example 4

Find the derivative of $y = \frac{1 + \sin(x)}{x + \cos(x)}$

$$y = \frac{1 + \sin(x)}{x + \cos(x)}$$

$$y' = \frac{(x + x \cos(x)) \frac{d}{dx}(1 + \sin(x)) - (1 + \sin(x)) \frac{d}{dx}(x + \cos(x))}{(x + \cos(x))^2}$$

$$y' = \frac{(x + x \cos(x))(\cos(x)) - (1 + \sin(x))(1 - \sin(x))}{(x + x \cos(x))^2}$$

$$y' = \frac{x \cos(x) + x \cos^2(x) - (1 - \sin^2(x))}{(x + x \cos(x))^2}$$

$$y' = \frac{x \cos(x) + x \cos^2(x) - 1 + \sin^2 x}{(x + x \cos(x))^2}$$

Example 5

Find the derivative of $y = \frac{1 + \sec(x)}{\tan(x)}$

$$y = \frac{1 + \sec(x)}{\tan(x)}$$

$$y' = \frac{\tan(x) \frac{d}{dx}(1 + \sec(x)) - (1 + \sec(x)) \frac{d}{dx} \tan x}{(\tan(x))^2}$$

$$y' = \frac{\tan(x)(\sec(x) \tan(x)) - (1 + \sec(x)) \sec^2(x)}{\tan^2 x}$$

$$y' = \frac{\sec(x) \tan^2(x) - \sec^2(x) - \sec^3(x)}{\tan^2(x)} = \frac{\sec(x) \tan^2(x) - \sec^3(x) - \sec^2(x)}{\tan^2(x)}$$

$$= \frac{\sec(x)(\tan^2(x) - \sec^2(x)) - \sec^2(x)}{\tan^2(x)} = \frac{\sec(x)(-1) - \sec^2(x)}{\tan^2(x)} = \frac{-\sec(x) - \sec^2(x)}{\tan(x)}$$

Example 6

Find the derivative of $y = e^x \cos x$

$$y = e^x \cos x$$

$$y' = \frac{d}{dx} e^x (\cos(x)) + \frac{d}{dx} (\cos(x)) e^x$$

$$y' = e^x \cos(x) - e^x \sin(x)$$

Example 8

Find the equation of a tangent line to the curve $y = \sec(x) - 2 \cos(x)$ at the point $\left(\frac{\pi}{3}, 1\right)$

$$y = \sec(x) - 2 \cos(x)$$

$$y' = \sec(x) \tan(x) - 2 \sin(x)$$

$$y' = \sec\left(\frac{\pi}{3}\right) \tan\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)$$

$$y'\left(\frac{\pi}{3}\right) = (2)(\sqrt{3}) - \frac{\sqrt{3}}{2} = \frac{4\sqrt{3} - \sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

$$y - 1 = \frac{3\sqrt{3}}{2} \left(x - \frac{\pi}{3}\right)$$

$$y = \frac{3\sqrt{3}}{2} \left(x - \frac{\pi}{3}\right) + 1$$
