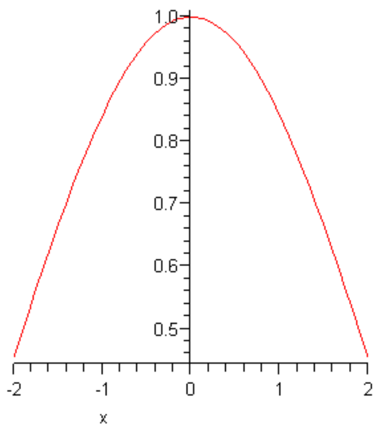
Math 151
Section 3.4

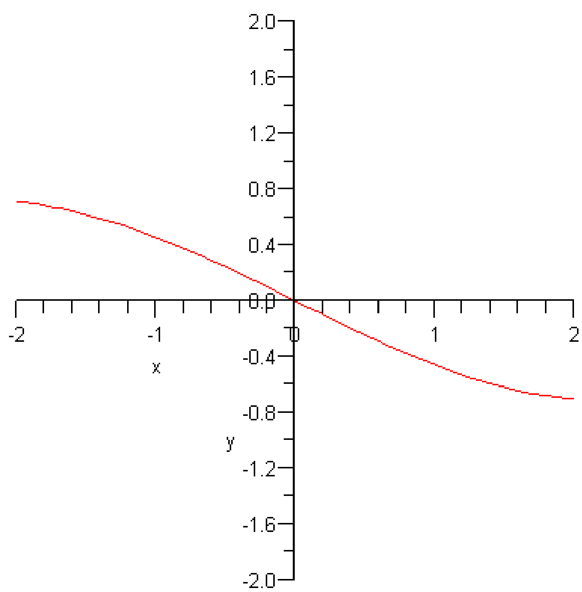
Derivatives of Trigonometric Functions

Review

$$\lim_{h\to 0}\frac{\sin(h)}{h}=1$$



$$\lim_{h\to 0}\frac{\cos(h)-1}{h}=0$$



The Derivative of Sine

$$
\begin{aligned}
f(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}\\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)-\sin(x)}{h}+\lim_{h\to 0}\frac{\cos(x)\sin(h)}{h}\\
&= \lim_{h\to 0}\frac{\sin(x)\left(\cos(h)-1\right)}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}\\
&= \sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}\\
&= \sin(x)(0)+\cos(x)(1)\\
&= \cos(x)
\end{aligned}
$$

Rule 1: *If* $f(x)=\sin(x),\ f'(x)=\cos(x)$

The Derivative of Cosine

$$
\begin{aligned}
f(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}\\
&= \lim_{h\to 0}\frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h}\\
&= \lim_{h\to 0}\frac{\cos(x)\cos(h)-\cos(x)}{h}-\lim_{h\to 0}\frac{\sin(x)\sin(h)}{h}\\
&= \lim_{h\to 0}\frac{\cos(x)\left(\cos(h)-1\right)}{h}-\sin(x)\lim_{h\to 0}\frac{\sin(h)}{h}\\
&= \cos(x)\lim_{h\to 0}\frac{\cos(h)-1}{h}+\sin(x)\lim_{h\to 0}\frac{\sin(h)}{h}\\
&= \cos(x)(0)-\sin(x)(1)\\
&= -\sin(x)
\end{aligned}
$$

Rule 2: *If* $f(x)=\cos(x),\ f'(x)=-\sin(x)$

The Derivative of Tangent

$If\ \ f(x) = \tan(x),\ then\ \ f(x) = \sec^2(x)$

Derivatives of the six trigonometric functions

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$
$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$
$$\frac{d}{dx}(\tan(x)) = \sec^2(x)$$
$$\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$$
$$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$$
$$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$$

Example 1

Find the derivative of $y = 2\csc(x) + 5\cos(x)$

$$y = 2\csc(x) + 5\cos(x)$$
$$y' = \frac{d}{dx}(2\csc(x)) + \frac{d}{dx}(5\cos(x))$$
$$y' = -2\csc(x)\cot(x) - 5\sin(x)$$

Example 2

Find the derivative of $y = \sqrt{x}\sin(x)$

$$y = \sqrt{x}\sin(x)$$
$$y = x^{\frac{1}{2}}\sin(x)$$
$$y' = \frac{d}{dx}\left(x^{\frac{1}{2}}\right)\sin(x) + \frac{d}{dx}(\sin(x))\left(x^{\frac{1}{2}}\right)$$
$$y' = \frac{1}{2}x^{\frac{1}{2}-1}\sin(x) + \cos(x)\left(x^{\frac{1}{2}}\right)$$
$$y' = \frac{1}{2}x^{-\frac{1}{2}}\sin(x) + x^{\frac{1}{2}}\cos(x)$$
$$y' = \frac{\sin(x)}{2\sqrt{x}} + \sqrt{x}\cos(x)$$

Example 3

Find the derivative of $y = e^{u}\left(\cos u + cu\right)$

$$y = e^{u}\left(\cos u + cu\right)$$
$$y' = \frac{d}{du}\left(e^{u}\right)\left(\cos u + cu\right) + \frac{d}{du}\left(\cos u + cu\right)e^{u}$$
$$y' = e^{u}\left(\cos u + cu\right) + \left(-\sin u + c\right)e^{u}$$
$$y' = e^{u}\cos u + cue^{u} - e^{u}\sin u - ce^{u}$$
$$y' = e^{u}\left(\cos u - \sin u\right) - ce^{u}\left(u - 1\right)$$

Example 4

Find the derivative of $y = \frac{1+\sin(x)}{x+\cos(x)}$

$$y = \frac{1+\sin(x)}{x+\cos(x)}$$

$$y' = \frac{\left(x + x\cos(x)\right)\frac{d}{dx}\left(1+\sin(x)\right) - \left(1+\sin(x)\right)\frac{d}{dx}\left(x+\cos(x)\right)}{\left(x+\cos(x)\right)^2}$$

$$y' = \frac{\left(x + x\cos(x)\right)\left(\cos(x)\right) - \left(1+\sin(x)\right)\left(1-\sin(x)\right)}{\left(x + x\cos(x)\right)^2}$$

$$y' = \frac{x\cos(x) + x\cos^2(x) - (1-\sin^2(x))}{\left(x + x\cos(x)\right)^2}$$

$$y' = \frac{x\cos(x) + x\cos^2(x) - 1 + \sin^2 x}{\left(x + x\cos(x)\right)^2}$$

Example 5

Find the derivative of $y = \frac{1+\sec(x)}{\tan(x)}$

$$y = \frac{1+\sec(x)}{\tan(x)}$$

$$y' = \frac{\tan(x)\frac{d}{dx}\left(1+\sec(x)\right) - \left(1+\sec(x)\right)\frac{d}{dx}\tan x}{(\tan(x))^2}$$

$$y' = \frac{\tan(x)(\sec(x)\tan(x)) - (1+\sec(x))\sec^2(x)}{\tan^2 x}$$

$$y' = \frac{\sec(x)\tan^2(x) - \sec^2(x) - \sec^3(x)}{\tan^2(x)} = \frac{\sec(x)\tan^2(x) - \sec^3(x) - \sec^2(x)}{tqn^2(x)}$$

$$= \frac{\sec(x)(\tan^2(x) - \sec^2(x)) - \sec^2(x)}{\tan^2(x)} = \frac{\sec(x)(-1) - \sec^2(x)}{\tan^2(x)} = \frac{-\sec(x) - \sec^2(x)}{\tan(x)}$$

Example 6

Find the derivative of $y = e^x \cos x$

$$y = e^x \cos x$$
$$y' = \frac{d}{dx}e^x(\cos(x)) + \frac{d}{dx}(\cos(x))e^x$$
$$y' = e^x \cos(x) - e^x \sin(x)$$

Example 8

Find the equation of a tangent line to the curve $y = \sec(x) - 2\cos(x)$ at the point $\left(\frac{\pi}{3}, 1\right)$

$$y = \sec(x) - 2\cos(x)$$
$$y' = \sec(x)\tan(x) - 2\sin(x)$$
$$y' = \sec\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)$$
$$y'\left(\frac{\pi}{3}\right) = (2)\left(\sqrt{3}\right) - \frac{\sqrt{3}}{2} = \frac{4\sqrt{3} - \sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

$$y - 1 = \frac{3\sqrt{3}}{2}(x - \frac{\pi}{3})$$
$$y = \frac{3\sqrt{3}}{2}\left(x - \frac{\pi}{2}\right) + 1$$