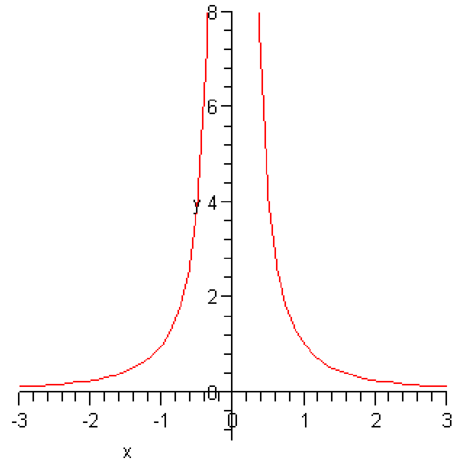
Section 2.5

Limits Involving Infinity

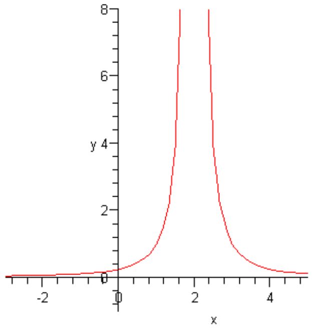
At the end of the section 2.2 we claimed that $\lim_{x\to 0}\frac{1}{x^2}$ did not exist. In this section, we will change the notation slightly by using the concept of infinity. Taking a closer look at the graph of $f(x)=\frac{1}{x^2}$, we will discover that functions approaches positive infinity on both sides of zero.



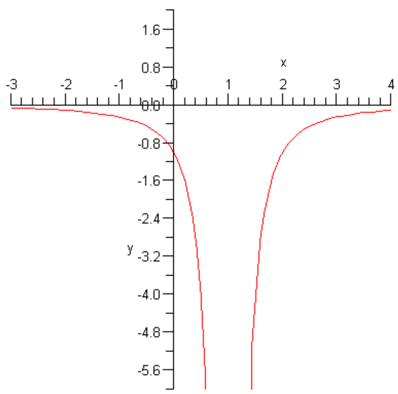
In this section, we describe the behavior by the notation: $\lim_{x\to 0}\frac{1}{x^2}=\infty$

Definition: The notation $\lim_{x\to a} f(x)=\infty$ means that the values of $f(x)$ can be arbitrarily large by taking x sufficiently close to a (on either side of a) but not equal to a.

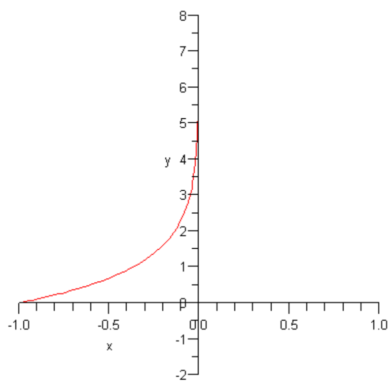
Definition: The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true.



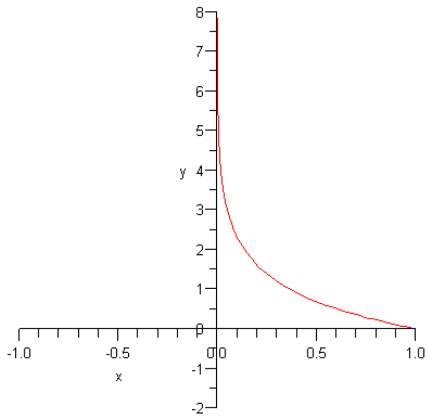
$$\lim_{x \to a} f(x) = \infty$$



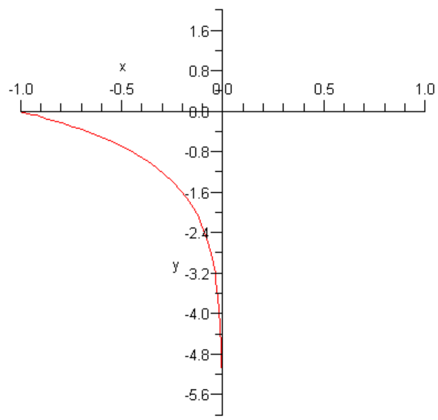
$$\lim_{x \to a} f(x) = -\infty$$



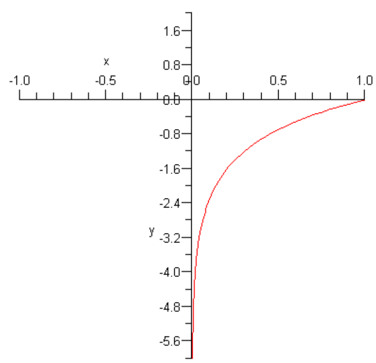
$$\lim_{x \to a^-} f(x) = \infty$$



$$\lim_{x\to a^+} f(x) = \infty$$



$$\lim_{x\to a^-} f(x) = -\infty$$



$$\lim_{x\to a^-} f(x) = -\infty$$

Example 1

Evaluate $\lim_{x\to\infty}\frac{3x^2-2x-6}{2x^2-4x+2}$

$$\lim_{x\to\infty}\frac{3x^2-2x-6}{2x^2-4x+2}$$

$$\lim_{x\to\infty}\frac{\frac{3x^2-2x-6}{x^2}}{\frac{2x^2-4x+2}{x^2}}$$

$$\lim_{x\to\infty}\frac{\frac{3x^2}{x^2}-\frac{2x}{x^2}-\frac{6}{x^2}}{\frac{2x^2}{x^2}-\frac{4x}{x^2}+\frac{2}{x^2}}$$

$$\lim_{x\to\infty}\frac{3-\frac{2}{x}-\frac{6}{x^2}}{2-\frac{4}{x}+\frac{2}{x^2}}$$

$$\frac{3-\frac{2}{\infty}-\frac{6}{\infty}}{2-\frac{4}{\infty}+\frac{2}{\infty}}$$

$$\frac{3-0-0}{2-0+0}$$

$$\frac{3}{2}$$

Example 2

Evaluate $\lim\limits_{x\to\infty}\dfrac{x^2-3}{x^3-x^2+2}$

$$\lim_{x\to\infty}\frac{x^2-3}{x^3-x^2+2}=\lim_{x\to\infty}\frac{\dfrac{x^2-3}{x^3}}{\dfrac{x^3-x^2+2}{x^3}}=\lim_{x\to\infty}\frac{\dfrac{x^2}{x^3}-\dfrac{3}{x^3}}{\dfrac{x^3}{x^3}-\dfrac{x^2}{x^3}+\dfrac{2}{x^3}}=\lim_{x\to\infty}\frac{\dfrac{1}{x}-\dfrac{3}{x^3}}{1-\dfrac{1}{x^2}+\dfrac{2}{x^3}}=\frac{\dfrac{1}{\infty}-\dfrac{3}{\infty}}{1-\dfrac{1}{\infty}+\dfrac{2}{\infty}}=\frac{0-0}{1-0+0}=0$$

Example 3

Evaluate $\lim\limits_{x\to\infty}\dfrac{x+4}{x-2}$

$$\lim_{x\to\infty}\frac{x+4}{x-2}=\lim_{x\to\infty}\frac{\dfrac{x}{x}+\dfrac{4}{x}}{\dfrac{x}{x}-\dfrac{2}{x}}=\lim_{x\to\infty}\frac{1+\dfrac{4}{x}}{1-\dfrac{2}{x}}=\frac{1+\dfrac{4}{\infty}}{1-\dfrac{2}{x}}=\frac{1+0}{1-0}=\frac{1}{1}=1$$

Example 4

Evaluate $\lim\limits_{x\to\infty}\left(\sqrt{4x^2+x}-2x\right)$

$$\lim_{x\to\infty}\left(\sqrt{4x^2+x}-2x\right)\frac{\sqrt{4x^2+x}+2x}{\sqrt{4x^2+x}+2x}=\lim_{x\to\infty}\frac{\left(\sqrt{4x^2+x}-2x\right)\left(\sqrt{4x^2+x}-2x\right)}{\sqrt{4x^2+x}+2x}$$

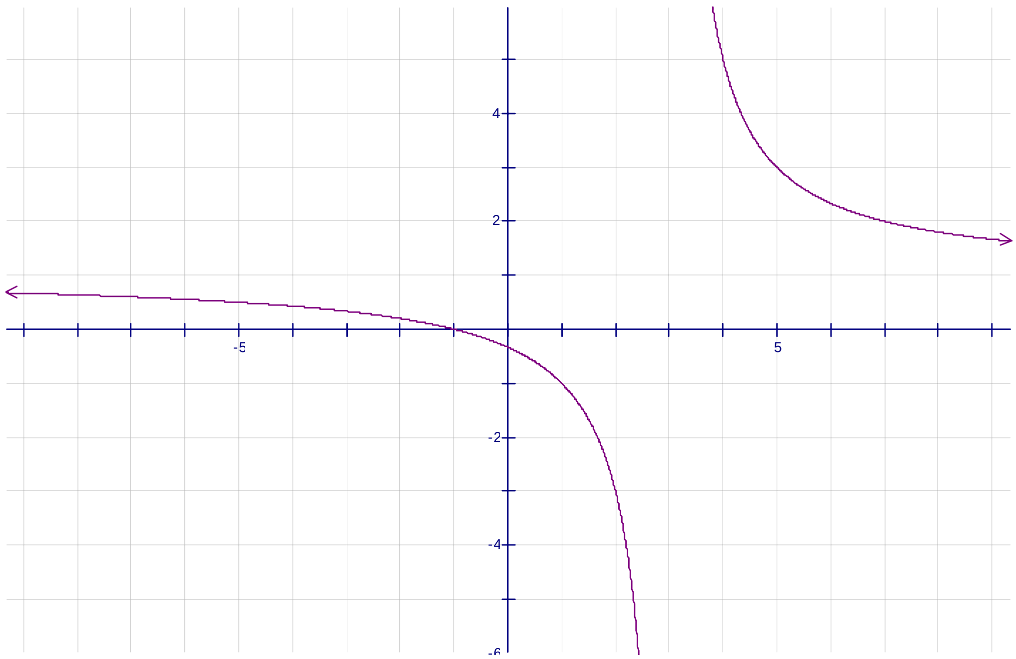
$$=\lim_{x\to\infty}\frac{\left(\sqrt{4x^2+x}\right)^2-2x\sqrt{4x^2+x}+2x\sqrt{4x^2+x}-4x^2}{\sqrt{4x^2+x}+2x}=\lim_{x\to\infty}\frac{4x^2+x-4x^2}{\sqrt{4x^2+x}+2x}=\lim_{x\to\infty}\frac{x}{\sqrt{4x^2+x}+2x}$$

$$=\lim_{x\to\infty}\frac{\dfrac{x}{x}}{\dfrac{\sqrt{4x^2+x}+2x}{x}}=\lim_{x\to\infty}\frac{1}{\sqrt{\dfrac{4x^2}{x^2}+\dfrac{x}{x^2}}+\dfrac{2x}{x}}=\lim_{x\to\infty}\frac{1}{\sqrt{4+\dfrac{1}{x}}+2}=\frac{1}{\sqrt{4+\dfrac{1}{\infty}}+2}=\frac{1}{\sqrt{4}+2}=\frac{1}{2+2}=\frac{1}{4}$$

Example 5

Find $\lim_{x \to 3} \frac{x+1}{x-3}$

Graph $f(x) = \frac{x+1}{x-3}$



$\lim_{x \to 3} \frac{x+1}{x-3}$ *Does not exist*