Section 2.4

Continuity

Definition: A function f is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a)$$

Notice that the definition implicitly requires three things if f is continuous at a number a:

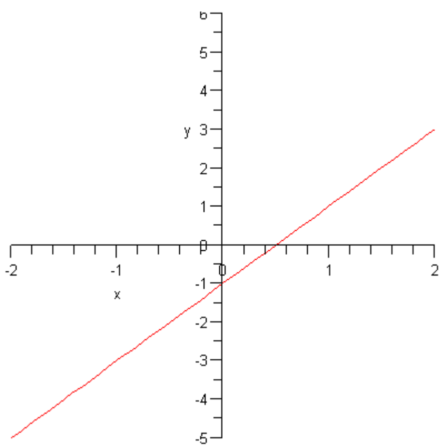
1) $f(a)$ *is defined*

2) $\lim_{x \to a} f(x)$ *exist*

3) $\lim_{x \to a} f(x) = f(a)$

Example 1

Discuss the continuity of $f(x)$ at $x = 1$



1) $f(1) = 1$, *so is defined at* $x = 1$

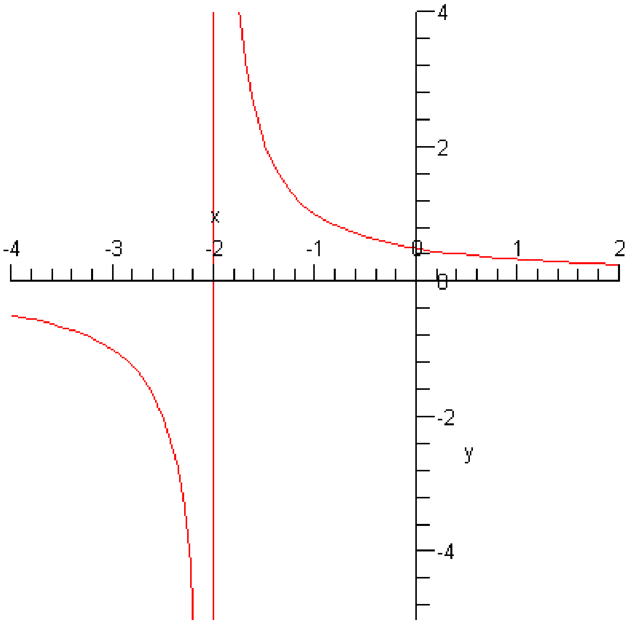
2) $\lim_{x \to 1} f(x) = 1$, so the limit exist at $x = 1$

3) $\lim_{x \to 1} f(x) = f(1) = 1$

Therefore, the function is continuous at $x = 1$.

Example 2

Discuss the continuity of $f(x)$ at $x = -2$



1) $f(-2)$ *is undefined*

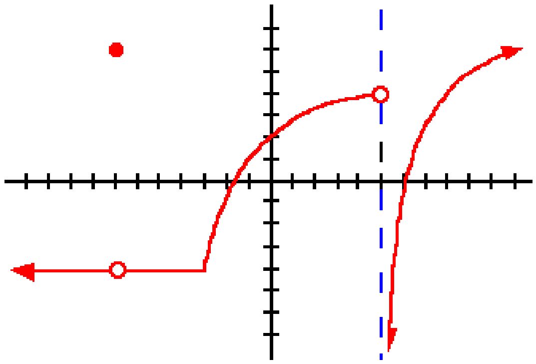
2) $\lim_{x \to -2} f(x)$ *does not exist*

3) $\lim_{x \to -2} f(x) \neq f(1)$

Therefore, the function is not continuous at $x = -2$.

Example 3

Discuss the continuity of $f(x)$ at $x=-6$ and $x=5$



Look $x=-6$

1) $f(-6)=6$ *is defined*

2) $\lim_{x\to -6} f(x)$ *exist*

3) $\lim_{x\to --6} f(x) \neq f(-6)$

Therefore, the function is not continuous at $x=-6$.

Look $x=5$

1) $f(5)$ *is undefined*

2) $\lim_{x\to 5} f(x)$ *exist*

3) $\lim_{x\to 5} f(x) \neq f(5)$

Therefore, the function is not continuous at $x=5$.

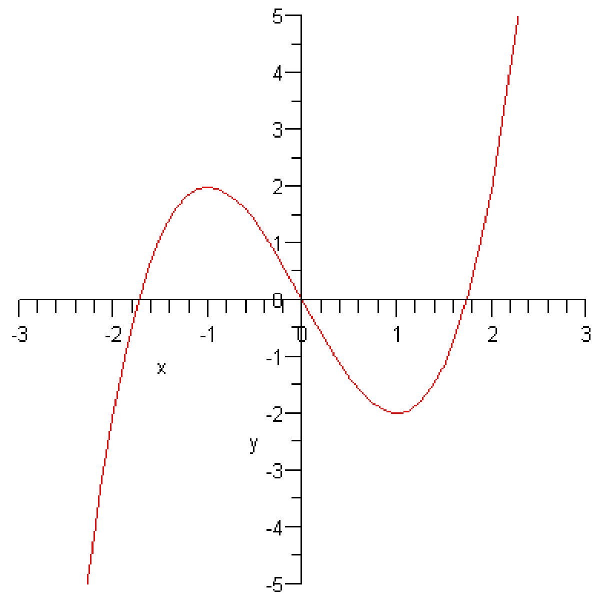
The graph of a continuous function is piecewise smooth.

Continuous Intervals

A function is continuous on an interval if the function is continuous at every point of the interval.

Example 4

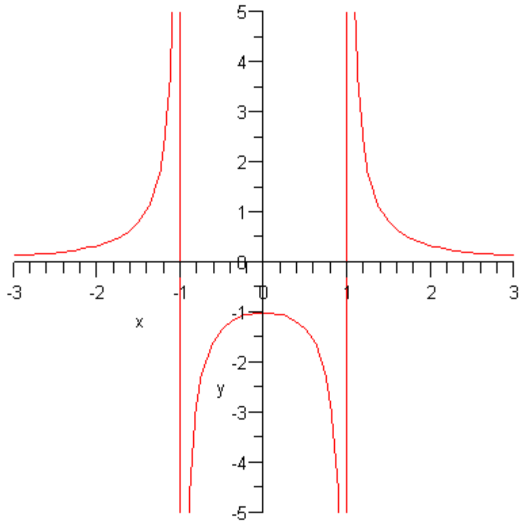
Discuss the continuity of the given function.



The graph is piece wise smooth, since it has breaks, holes, or asymptotes. Therefore, the function is continuous on $(-\infty, \infty)$

Example 5

Discuss the continuity of the function.



The function is discontinuous at $x = -1$ and $x = -1$
The function is continuous on $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

Theorem: Polynomial functions are continuous everywhere.

Theorem: Rational functions are continuous on their domain

Example 6

Give the intervals where the function is continuous.

$f(x) = x^3 + 2x^2$

Continuous on $(-\infty, \infty)$

Example 7

Give the intervals where the function is continuous.

$$f(x) = \frac{1}{x^2 - 4}$$

$$f(x) = \frac{1}{x^2 - 4} = \frac{1}{(x-2)(x+2)} \Rightarrow f \ \textit{is undefined at} \ x = -2 \ \textit{and} \ x = 2$$

Continuous on $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

Example 8

Give the intervals where the function is continuous.

$$f(x) = \frac{x-3}{x^2 + x - 12}$$

$$f(x) = \frac{x-3}{x^2 + x - 12} = \frac{x-3}{(x-3)(x+4)} \Rightarrow f \ \textit{is undefined at} \ x = 3 \ \textit{and} \ x = -4$$

Continuous on $(-\infty, -4) \cup (-4, 3) \cup (3, \infty)$

Example 9

Give the intervals where the function is continuous.

$$f(x) = x - \cos x$$

Continuous on $(-\infty, \infty)$

Example 10: Give the intervals where the function is continuous.

$$f(x) = x + e^{x+3}$$

Solution: Continuous on $(-\infty, \infty)$

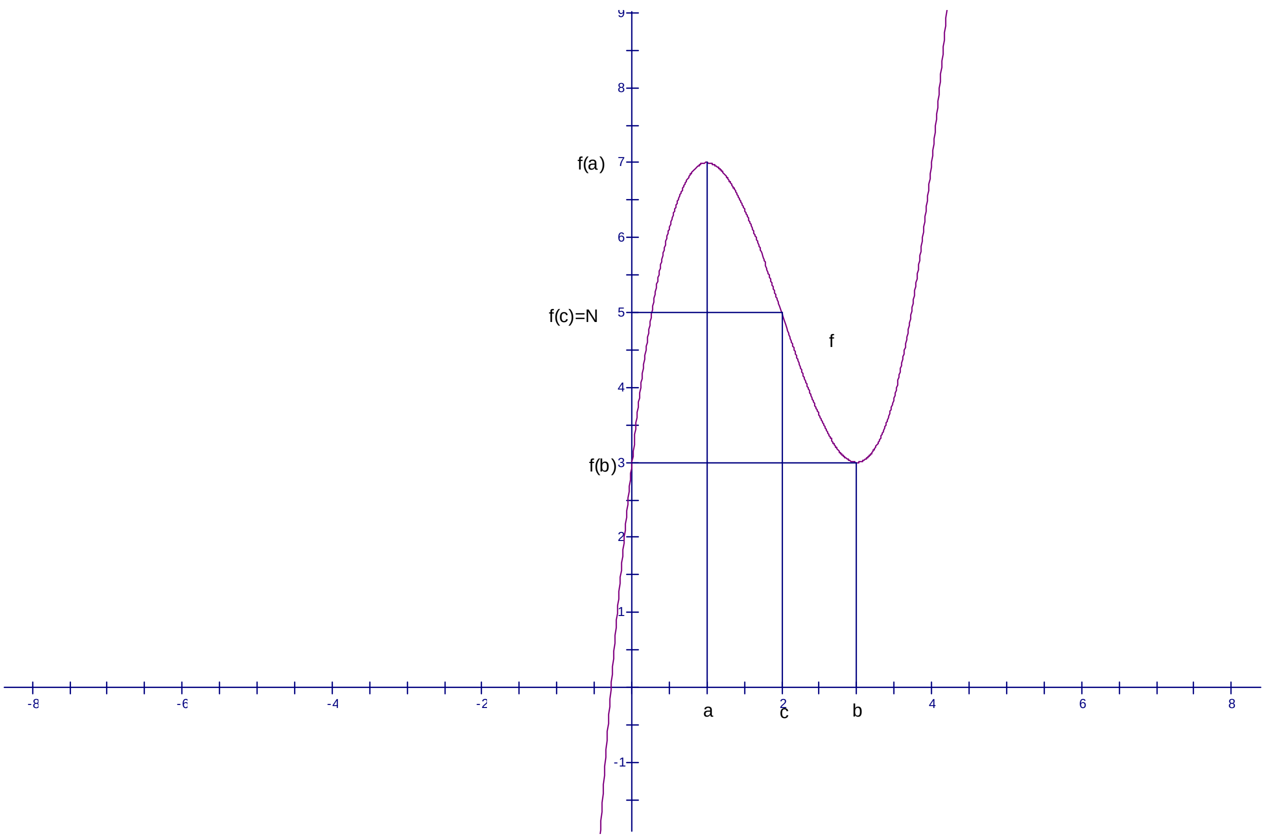
Theorem

If b is a real number and f and g are continuous at $x = c$, then the following are also continuous at c.

1) Scalar multiple: bf
2) Sum: $f + g$
3) Difference: $f - g$
4) Product: fg
5) Quotient: $\frac{f}{g}, if \; g(c) \neq 0$

Intermediate Value Theorem

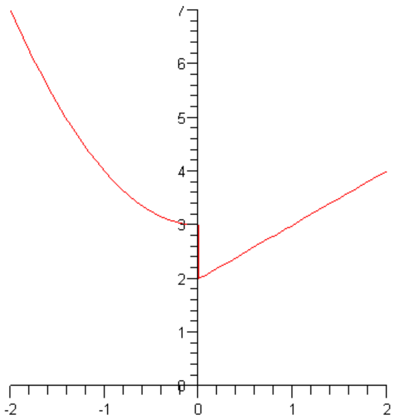
Suppose that f is continuous on the closed interval $[a,b]$ and let N be any real number between $f(a)$ and $f(b)$. Then there exist a number c in (a,b) such that $f(c) = N$



Example 11

Give the intervals where the function is continuous.

$$f(x) = \begin{cases} x^2 + 3 & x < 0 \\ x + 2 & x \geq 0 \end{cases}$$



The function is discontinuous at $x = 0$
Function is continuous on $(-\infty, \infty)$

Exercises

Give the intervals where the function is continuous

1) $f(x) = x^2 - x + 1$

2) $f(x) = \frac{1}{x^2 + 1}$

3) $f(x) = x + \sin x$

4) $f(x) = \cos\left(\frac{\pi x}{2}\right)$

5) $f(x) = \frac{1}{x + 1}$

6) $f(x) = \frac{x - 3}{x^2 - 9}$

7) $f(x) = \frac{x + 2}{x^2 - 3x - 10}$

8) $f(x) = e^{x-1}$

9) $f(x) = e^x + 1$

10) $f(x) = \ln(x - 4)$