Section 2.2/2.3

Limits

Definition of a Limit

$$\lim_{x \to a} f(x) = L$$

The limit of $f(x)$, as x approaches a, equals L

We can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a (on either side of a) but not equal to a.

Evaluating limits using the graph of a function

Example 1

Find the limit: $\lim_{x \to 1} x^2 + 1$

Find the limit of the function as x approaches 1

x	.9	.99	.999	1.1	1.01	1.001
$f(x)$	1.810	1.980	1.998	2.210	2.020	2.002

The values in the table seem to approach 2 as x approaches 1. There the limit of function as x approaches 1 is 2. This is written symbolically as $\lim_{x \to 1} x^2 + 1 = 2$

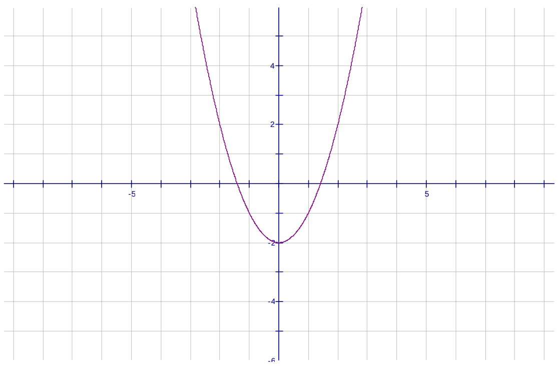
Example 2

Using the graph below evaluate each limit.

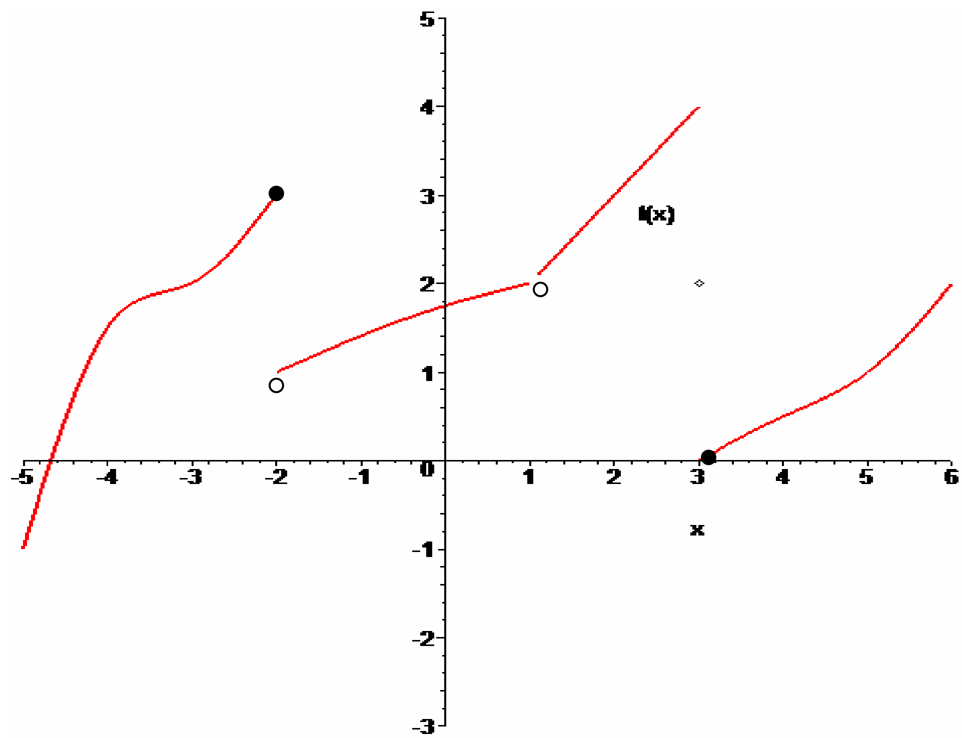
a) Find $\lim_{x \to 2} f(x)$: **Solution** $\lim_{x \to 2} f(x) = 2$

b) Find $\lim_{x \to -0} f(x)$: **Solution** $\lim_{x \to 0} f(x) = -2$

c) Find $\lim_{x \to -2} f(x)$: **Solution** $\lim_{x \to -2} f(x) = 2$



Example 3



a) Find $\lim_{x \to -2^-} f(x)$: **Solution** $\lim_{x \to -2^-} f(x) = 3$

b) Find $\lim_{x \to -2^+} f(x)$: **Solution** $\lim_{x \to -2^-} f(x) = 1$

c) Find $\lim_{x \to -2} f(x)$: **Solution** $\lim_{x \to -2} f(x)\ DNE$

d) Find $\lim_{x \to 1^-} f(x)$: **Solution** $\lim_{x \to -1^-} f(x) = 2$

e) Find $\lim_{x \to 1^+} f(x)$: **Solution** $\lim_{x \to -1^+} f(x) = 2$

f) Find $\lim_{x \to 3^-} f(x)$: **Solution** $\lim_{x \to 3^+} f(x) = 4$

g) Find $\lim_{x \to 3^+} f(x)$: **Solution** $\lim_{x \to 3^+} f(x) = 0$

h) Find $\lim_{x \to 3} f(x)$: **Solution** $\lim_{x \to 3^+} f(x)\ DNE$

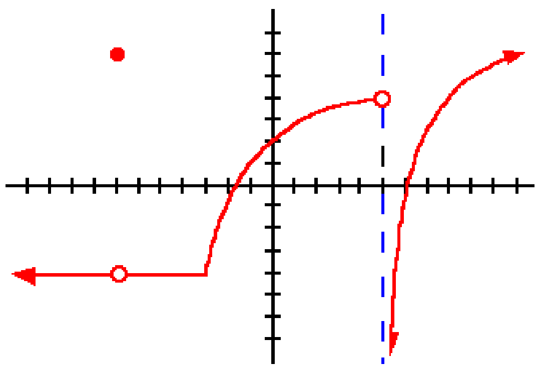
Example 4

Using the graph below evaluate each limit.

a) Find $\lim_{x \to 5} f(x)$: **Solution** $\lim_{x \to 5} f(x) = Does\ Not\ Exist$

b) Find $\lim_{x \to -7} f(x)$: **Solution** $\lim_{x \to -7} f(x) = -4$

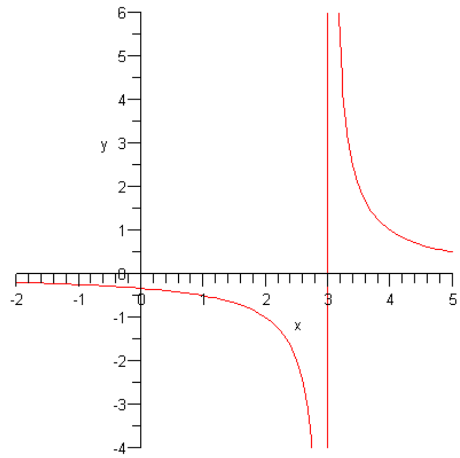
c) Find $\lim_{x \to 0} f(x)$: **Solution** $\lim_{x \to 0} f(x) = 2$



Example 5

Using the graph below evaluate each limit. Find $\lim_{x \to 3} f(x)$:

Solution $\lim_{x \to 3} f(x) = Does\ Not\ Exist$



Recall from example 1: $\lim_{x \to 1} x^2 + 1 = 2$

Definition 1: $\lim_{x \to c} f(x) = f(c)$

Thus the limit in the above example can be found by using definition 1,
$\lim_{x \to 1} f(x) = f(1) = 1^2 + 1 = 1 + 1 = 2$

Example 6

Find $\lim_{x \to 3} x^3$

Solution: $\lim_{x \to 3} x^3 = 3^3 = 27$

Example 7

Find $\lim_{x \to -4} x^2 + 5x$

Solution: $\lim_{x \to -4} x^2 + 5x = (-4)^2 + 5(-4) = 16 - 20 = -4$

Example 8

Find $\lim_{x \to 3} \frac{5}{x-3}$

Solution: $\lim_{x \to 3} \frac{5}{x-3} = \frac{5}{3-3} = \frac{5}{0}$ *Undefined*

Limit does not exist

Example 9

Find $\lim_{x\to1}\sqrt{2x-5}$

Solution: $\lim_{x\to1}\sqrt{2x-5}=\sqrt{2(1)-5}=\sqrt{2-5}=\sqrt{-3}\ \mathit{Undefined}$

Limit does not exist

Example 10

Find $\lim_{x\to3}\dfrac{x-3}{x^2-9}$

Solution: $\lim_{x\to3}\dfrac{x-3}{x^2-9}=\lim_{x\to3}\dfrac{x-3}{(x-3)(x+3)}=\lim_{x\to3}\dfrac{1}{x+3}=\dfrac{1}{3+3}=\dfrac{1}{6}$

In example 5, if you substitute 3 in for x before you reduce the polynomial down, you will get an undetermined form. $\dfrac{0}{0}$

Example 11

Find $\lim_{x\to2}\dfrac{2-x}{x^2-4}$

Solution: $\lim_{x\to2}\dfrac{2-x}{x^2-4}=\lim_{x\to3}\dfrac{-(x-2)}{(x-2)(x+2)}=\lim_{x\to3}\dfrac{-1}{x+2}=\dfrac{-1}{2+2}=-\dfrac{1}{4}$

Example 12

Find $\lim_{t\to1}\dfrac{t^2+t-2}{t^2-1}$

Solution: $\lim_{t\to1}\dfrac{t^2+t-2}{t^2-1}=\lim_{t\to1}\dfrac{(t+2)(t-1)}{(t-1)(t+1)}=\lim_{t\to3}\dfrac{t+2}{t+1}=\dfrac{1+2}{1+1}=\dfrac{3}{2}$

Example 13

Find $\lim_{x \to -1} \frac{4x-5}{3-x}$

Solution: $\lim_{x \to -1} \frac{4x-5}{3-x} = \frac{4(-1)-5}{3-(-1)} = \frac{-4-5}{4} = -\frac{9}{4}$

Example 14

Find $\lim_{x \to 2} e^{x-2}$

Solution: $\lim_{x \to 2} e^{x-2} = e^{2-2} = e^{0} = 1$

Example 15

Find $\lim_{x \to \frac{\pi}{2}} \sin x$

Solution: $\lim_{x \to \frac{\pi}{2}} \sin x = \sin\left(\frac{\pi}{2}\right) = 1$

Example 16

Find $\lim_{h \to 0} \frac{(h+1)^2 - 1}{h}$

Solution:

$$\lim_{h \to 0} \frac{(h+1)^2 - 1}{h} = \lim_{h \to 0} \frac{(h+1)(h+1) - 1}{h} = \lim_{h \to 0} \frac{h^2 + h + h + 1 - 1}{h} = \lim_{h \to 0} \frac{h^2 + 2h}{h} = \lim_{h \to 0} \frac{h(h+2)}{h}$$

$$= \lim_{h \to 0} h + 2 = 0 + 2 = 2$$

Example 17

Find $\lim_{th\to 0}\frac{\sqrt{t^2+9}-3}{t^2}$

$$\lim_{th\to 0}\frac{\sqrt{t^2+9}-3}{t^2}=\lim_{h\to 0}\frac{\left(\sqrt{t^2+9}-3\right)\left(\sqrt{t^2+9}+3\right)}{t^2\left(\sqrt{t^2+9}+3\right)}=\lim_{t\to 0}\frac{\left(\sqrt{t^2+9}\right)^2+3\sqrt{t^2+9}-3\sqrt{t^2+9}-9}{t^2\left(\sqrt{t^2+9}+3\right)}$$

$$=\lim_{t\to 0}\frac{t^2+9-9}{t^2\left(\sqrt{t^2+9}+3\right)}=\lim_{t\to 0}\frac{t^2}{t^2\left(\sqrt{t^2+9}+3\right)}=\lim_{t\to 0}\frac{1}{\left(\sqrt{t^2+9}+3\right)}=\frac{1}{\sqrt{0^2+9}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{3+3}=\frac{1}{6}$$

Limit Laws

1) $\lim_{x\to a}\left(f(x)+g(x)\right)=\lim_{x\to a}f(x)+\lim_{x\to a}g(x)$

2) $\lim_{x\to a}\left(f(x)-g(x)\right)=\lim_{x\to a}f(x)-\lim_{x\to a}g(x)$

3) $\lim_{x\to a}\left(f(x)\cdot g(x)\right)=\lim_{x\to a}f(x)\cdot\lim_{x\to a}g(x)$

4) $\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right)=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}$

5) $\lim_{x\to a}c\cdot f(x)=c\cdot\lim_{x\to a}f(x)$

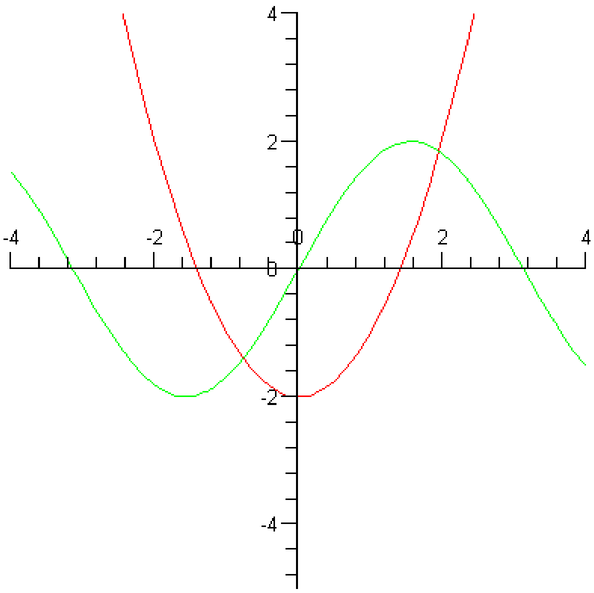
Example 18

Find $\lim_{x\to 0}\frac{\sqrt{5-x}-\sqrt{5}}{x}$

$$\lim_{x\to 0}\frac{\sqrt{5-x}-\sqrt{5}}{x}=\lim_{x\to 0}\frac{\sqrt{5-x}-\sqrt{x}}{x}\cdot\frac{\sqrt{5-x}+\sqrt{5}}{\sqrt{5-x}+\sqrt{5}}=\lim_{x\to 0}\frac{\left(\sqrt{5-x}\right)^2-\sqrt{5}\sqrt{5-x}+\sqrt{5}\sqrt{5-x}-5}{x\left(\sqrt{5-x}+\sqrt{5}\right)}$$

$$=\lim_{x\to 0}\frac{5-x-5}{x\left(\sqrt{5-x}+\sqrt{5}\right)}=\lim_{x\to 0}\frac{-x}{x\left(\sqrt{5-x}+\sqrt{5}\right)}=\lim_{x\to 0}\frac{-1}{\sqrt{5-x}+\sqrt{5}}=\frac{-1}{\sqrt{5-0}+\sqrt{5}}=\frac{-1}{2\sqrt{5}}$$

Example 19



$f(x) - green\ line$

$g(x) - red\ line$

Use to the above graph to evaluate each limit

1) Find $\lim_{x\to 0}\left(f(x)+g(x)\right)$

$$\lim_{x\to 0}\left(f(x)+g(x)\right)=\lim_{x\to 0}f(x)+\lim_{x\to 0}g(x)=0+2$$

2) Find $\lim_{x\to 0}\left(f(x)-g(x)\right)$

$$\lim_{x\to 0}\left(f(x)-g(x)\right)=\lim_{x\to 0}f(x)-\lim_{x\to 0}g(x)=0-2=-2$$

3) Find $\lim_{x\to 0}\left(f(x)\cdot g(x)\right)$

$$\lim_{x\to 0}\left(f(x)\cdot g(x)\right)=\lim_{x\to 0}f(x)\cdot\lim_{x\to 0}g(x)=0\cdot 2=0$$

4) $\lim_{x\to 0}\left(\frac{f(x)}{g(x)}\right)$

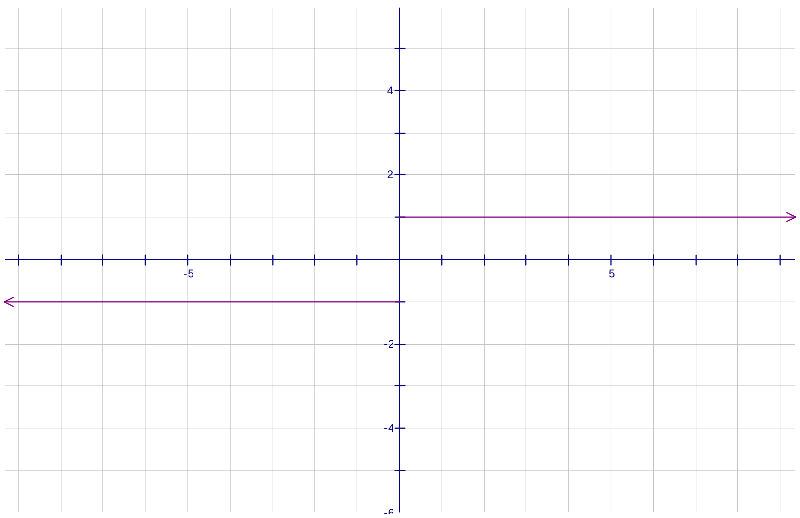
$$\lim_{x\to 0}\left(\frac{f(x)}{g(x)}\right)=\frac{\lim_{x\to 0}f(x)}{\lim_{x\to 0}g(x)}=\frac{0}{2}=0$$

Example 18

Evaluate $\lim_{x \to 2} \left(2x^3 + |2x+3| \right)$

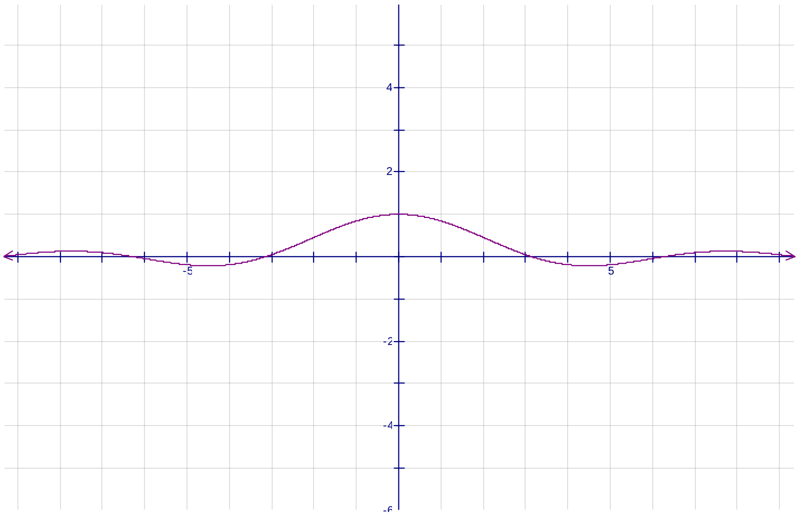
$$\lim_{x \to 2} \left(2x^3 + |2x+3| \right) = \lim_{x \to 2} 2x^3 + \lim_{x \to 2} |2x+3| = 2\left(2^3\right) + |2(3)+3| = 2(8) + |6+3| = 16 + |9| = 16 + 9 = 25$$

Example 20 Find $\lim_{x \to 0} \frac{|x|}{x}$



$\lim_{x \to 0} \frac{|x|}{x}$ *does not exist*

Example 21 Find $\lim_{x \to 0} \frac{\sin(x)}{x}$



$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$