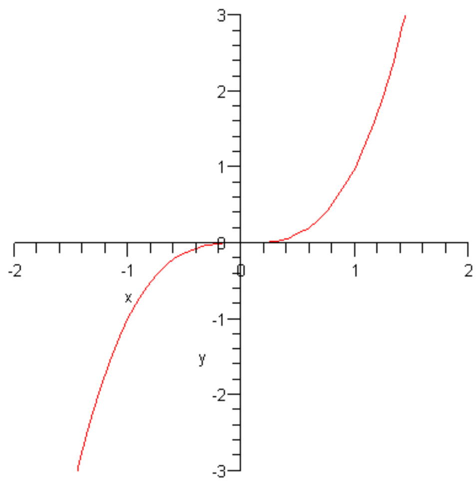
Section 1.6
Inverse Functions and Logarithms

One-to-one functions

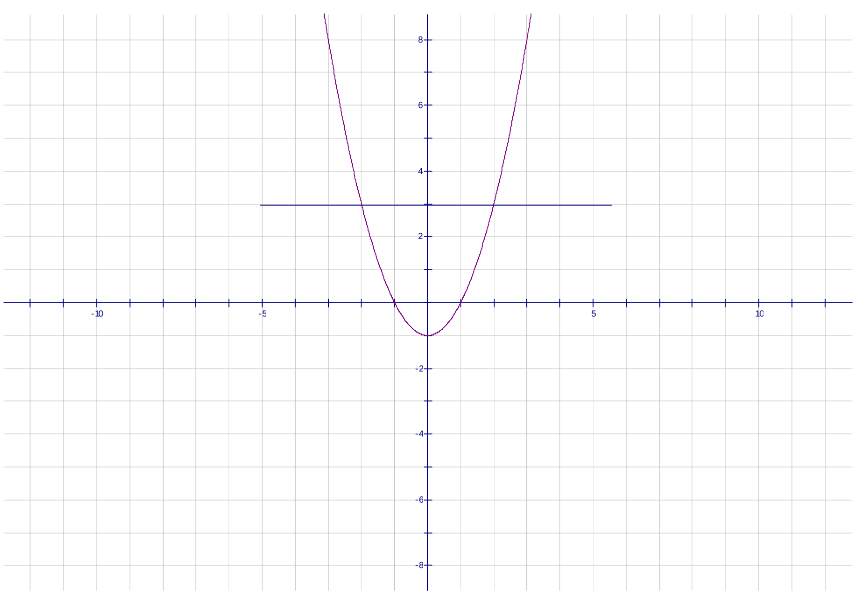
Definition: A function f is an one-to-one function if it never take one the same value twice. $If \; f(x_1) = f(x_2), then \; x_2 = x_1$

Horizontal Line Test: A function is one-to-one if and only no horizontal line intersects its graph more than once.

Graph of a one-to-one function



Graph of a function that is not one-to-one



If a function is one-to-one, then it has an inverse

Inverse functions

$$f^{-1}(x) = y \Leftrightarrow f(x) = y$$
$$domain\ of\ f^{-1} = range\ of\ f$$
$$range\ of\ f^{-1} = domain\ of\ f$$

The relationship between the inverse function and the composition

$$f(f^{-1}(x)) = x\ and\ f^{-1}(f(x)) = x$$

Example 1

Find the inverse of the function and show that $f(f^{-1}(x)) = x\ and\ f^{-1}(f(x)) = x$ given $f(x) = 2x - 3$

First, find the inverse.

$$f(x) = 2x - 3$$
$$y = 2x - 3$$
$$x = 2y - 3$$
$$x + 3 = 2y$$
$$\frac{x+3}{2} = y \Rightarrow f(x) = \frac{x+3}{2}$$

$$f(f^{-1}(x)) = f\left(\frac{x+3}{2}\right) = 2\left(\frac{x+3}{2}\right) - 3 = x + 3 - 3 = x$$

$$f^{-1}(f(x)) = f^{-1}(2x-3) = \frac{2x-3+3}{2} = \frac{2x}{2} = x$$

Example 2

Find the inverse of the function and show that $f(f^{-1}(x)) = x$ *and* $f^{-1}(f(x)) = x$ given $f(x) = x^3 - 3$

$$f(x) = x^3 - 3$$
$$y = x^3 - 3$$
$$x = y^3 - 3$$
$$x + 3 = y^3$$
$$\sqrt[3]{x+3} = \sqrt[3]{y^3}$$
$$\sqrt[3]{x+3} = y$$
$$f^{-1}(x) = \sqrt[3]{x+3}$$

$$f(f^{-1}(x)) = f(\sqrt[3]{x+3}) = \left(\sqrt[3]{x+3}\right)^3 - 3 = x + 3 - 3 = x$$

$$f^{-1}(f(x)) = f^{-1}(x^3 - 3) = \sqrt[3]{x - 3 + 3} = \sqrt[3]{x^3} = x$$

Basic Logarithms

Definition of a logarithm

$$\log_a x = y \Leftrightarrow a^y = x$$

Example 3

a) Write $\log_4 64 = 3$ as an exponential expression.

$$\log_4 64 = 3 \Rightarrow 4^3 = 64$$

b) Write $\log_5\left(\frac{1}{25}\right) = -2$

$$\log_5\left(\frac{1}{25}\right) = -2 \Rightarrow 5^{-2} = \frac{1}{25}$$

Example 4

a) Find $\log_2 16$

$$\log_2 16 = x$$
$$2^x = 16$$
$$2^x = 2^4$$
$$x = 4$$

b) Find $\log_4 \frac{1}{64}$

$$\log_4 \frac{1}{64} = x$$
$$4^x = \frac{1}{64}$$
$$4^x = \frac{1}{4^3}$$
$$4^x = 4^{-3}$$
$$x = -3$$

Rules for Logarithms

1) $\log_a (xy) = \log_a x + \log_a y$

2) $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$

3) $\log_a (x^r) = r \log_a x \quad (\textit{where is any real number})$

Example 5

Simplify $\log_2 20 - \log_2 5$

$$\log_2 20 - \log_2 5 = \log_2 \left(\frac{20}{5}\right) = \log_2 4 = 2$$

Example 6

Simplify $\log_4 8 + \log_4 8$

$$\log_4 8 + \log_4 8 = \log_4 (8 \cdot 8) = \log_4 64 = 3$$

Change of base formula.

$$\log_b a = \frac{\log a}{\log b}$$

Example 7

Evaluate $\log_6 40 = \frac{\log 40}{\log 6} = \frac{1.6021}{.7882} = 2.033$

The Natural Logarithm

$$\log_e x = y \Rightarrow e^y = x$$

Basic natural logarithm rules

1) $\ln x + \ln y = \ln(xy)$

2) $\ln x - \ln y = \ln\left(\frac{x}{y}\right)$

3) $\ln\left(x^a\right) = a \ln x$

4) $e^{\ln x} = x$ *and* $\ln\left(e^x\right) = x$

Example 8

Simplify $\ln 4 + 3\ln 2$

$$\ln 4 + 3\ln 2 = \ln 4 + \ln 2^3 = \ln 4 + \ln 8 = \ln(4 \cdot 8) = \ln 32$$

Example 9

Find $\log_4 256$

$$\log_4 256 = x \Rightarrow 4^x = 64 \Rightarrow 4^x = 4^3 \Rightarrow x = 3$$

Example 10

Find $\log_5 \frac{1}{125}$

$$\log_5 \frac{1}{125} = x \Rightarrow 5^x = \frac{1}{125} \Rightarrow 5^x = \frac{1}{5^5} \Rightarrow 5^x = 5^{-3} \Rightarrow x = -3$$

Note: $\log(100)$ is understood to be $\log_{10}(100)$

Therefore, you solve this logarithm as follows:

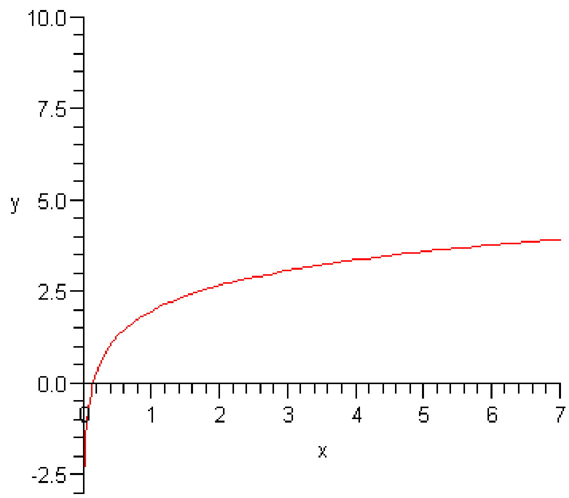
$$\log_{10}(100) = x \Rightarrow 10^x = 100 \Rightarrow 10^x = 10^2 \Rightarrow x = 2$$

Graphs of the natural log and log functions

Example 11

Graph $f(x) = \log(x) + 2$

x	$f(x) = \log(x) + 2$
1	$f(1) = \log(1) + 2 = 0 + 2 = 2$
10	$f(10) = \log(10) + 2 = 1 + 2 = 3$
20	$f(20) = \log(20) + 2 = 1.3 + 2 = 3.3$
30	$f(30) = \log(30) + 2 = 1.5 + 2 = 3.5$
100	$f(100) = \log(100) + 2 = 3 + 2 = 5$

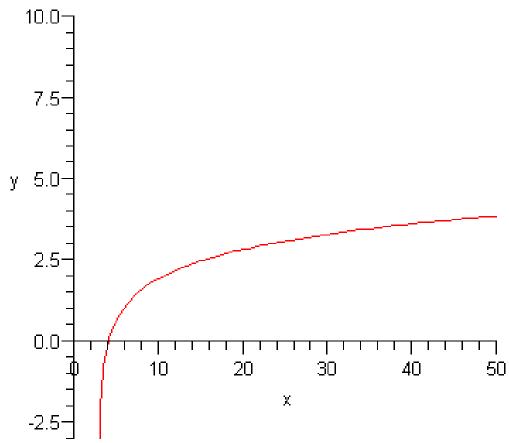


Domain: $(0, \infty)$
Range: $(-\infty, \infty)$

Example 12

Graph $f(x) = \log(x-3)$

x	$f(x) = \log(x-3)$
4	$f(4) = \log(4-3) = \log(1) = 0$
10	$f(10) = \log(10-3) = \log(7) = .85$
20	$f(20) = \log(20-3) = \log(17) = 1.23$
30	$f(30) = \log(30-3) = \log(27) = 1.43$
100	$f(100) = \log(100-3) = \log(97) = 1.99$



Domain: $(3, \infty)$
Range: $(-\infty, \infty)$

Example 14

Solve $e^{2x} = 8$

$$e^{2x} = 8$$
$$\ln\left(e^{2x}\right) = \ln 8$$
$$(2x)\ln e = \ln 8$$
$$2x = \frac{\ln 8}{\ln e}$$
$$2x = \frac{\ln 8}{\ln e}$$
$$\frac{1}{2}\cdot 2x = \frac{1}{2}\cdot\frac{\ln 8}{\ln e}$$
$$x = \frac{\ln 8}{2\ln e}$$

Example 15

Solve $3\ln(x+1) = 2$

$$3\ln(x+1) = 2$$
$$e^{3\ln(x+1)} = e^2$$
$$3e^{\ln(x+1)} = e^2$$
$$3(x+1) = e^2$$
$$3x + 3 = e^2$$
$$3x = e^2 - 3$$
$$x = \frac{e^2 - 3}{3}$$

Example 16

Solve $e^{x-3} = 5$

$e^{x-3} = 5$

$\ln\left(e^{x-3}\right) = \ln 5$

$(x-3)\ln e = \ln 5$

$(x-3)(1) = \ln 5$

$x - 3 = \ln 5$

$x = \ln 5 + 3$

$x \approx 4.61$

Example 17

If a bacteria population starts with 100 bacteria and double every 30 minutes, then the number of bacteria after t hours is $B(t) = 100(2)^{2t}$. How much time will it take for the population to grow to 30,000 bacteria?

$B(t) = 100(2)^{2t}$

$30000 = 100(2)^{2t}$

$\frac{30000}{100} = \frac{100(2)^{2t}}{100}$

$30 = (2)^{2t}$

$\log(300) = \log\left((2)^{2t}\right)$

$\log(300) = 2t\log(2)$

$2t = \frac{\log(300)}{\log(2)}$

$t = \frac{\log(300)}{2\log(2)}$

$t \approx 4.11\ hours$

Example 18

Find the inverse of $f(x) = e^{\sqrt{x}}$

$$f(x) = e^{\sqrt{x}}$$
$$y = e^{\sqrt{x}}$$
$$x = e^{\sqrt{y}}$$
$$\ln x = \ln\left(e^{\sqrt{y}}\right)$$
$$\ln x = \sqrt{y}$$
$$(\ln x)^2 = \left(\sqrt{y}\right)^2$$
$$y = (\ln x)^2$$

Example 19

If a bacteria population starts with 200 bacteria and double every three hours, then the number of bacteria after t hours is $B(t) = 200(2)^{\frac{t}{3}}$

a) Find the inverse of the function.

$$B(t) = 200(2)^{\frac{t}{3}}$$

$$y = 200(2)^{\frac{t}{3}}$$

$$t = 200(2)^{\frac{y}{3}}$$

$$\frac{t}{200} = \frac{200(2)^{\frac{y}{3}}}{200}$$

$$\frac{t}{200} = 2^{\frac{y}{3}}$$

$$\log\left(\frac{t}{200}\right) = \log\left(2^{\frac{y}{3}}\right)$$

$$\log t - \log(200) = \frac{y}{3}\log(2)$$

$$\frac{y}{3} = \frac{\log t - \log(200)}{\log(2)}$$

$$y = \frac{3(\log t - \log(200)}{\log(2)}$$

$$B^{-1}(t) = \frac{3(\log t - \log(200))}{\log(2)}$$

b) When will the population reach 10,000 bacteria

$$B^{-1}(10{,}000) = \frac{3(\log 10000 - \log(200))}{\log(2)} = \frac{3(4 - 2.30)}{.301} = \frac{3(1.7)}{.301} = \frac{5.1}{.301} \approx 17 \ days$$