

## Section 1.5 (Limits)

### Definition of a Limit

$$\lim_{x \rightarrow a} f(x) = L$$

The limit of  $f(x)$ , as  $x$  approaches  $a$ , equals  $L$

We can make the values of  $f(x)$  arbitrarily close to  $L$  by taking  $x$  to be sufficiently close to  $a$  (on either side of  $a$ ) but not equal to  $a$ .

Evaluating limits using the graph of a function

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### Approximating Limits

$$\lim_{x \rightarrow 1} x^2 + 1$$

$x$	.9	.99	.999	1.1	1.01	1.001
$f(x)$	$f(.9)$	$f(.9)$	$f(.9)$	$f(.9)$	$f(1.01)$	$f(1.001)$
	$= .9^2 + 1$	$= .99^2 + 1$	$= .999^2 + 1$	$= 1.1^2 + 1$	$= 1.01^2 + 1$	$= 1.001^2 + 1$
	$= 1.81$	$= 1.98$	$= 1.998$	$= 2.21$	$= 2.02$	$= 2.002$

The limit as  $x$  approaches from the right or “Right hand limit”

$$\lim_{x \rightarrow 1^+} x^2 + 1 = 2$$

The limit as  $x$  approaches from the left or “Left hand limit”

$$\lim_{x \rightarrow 1^-} x^2 + 1 = 2$$

The regular limit

$$\lim_{x \rightarrow 1} x^2 + 1 = 2$$

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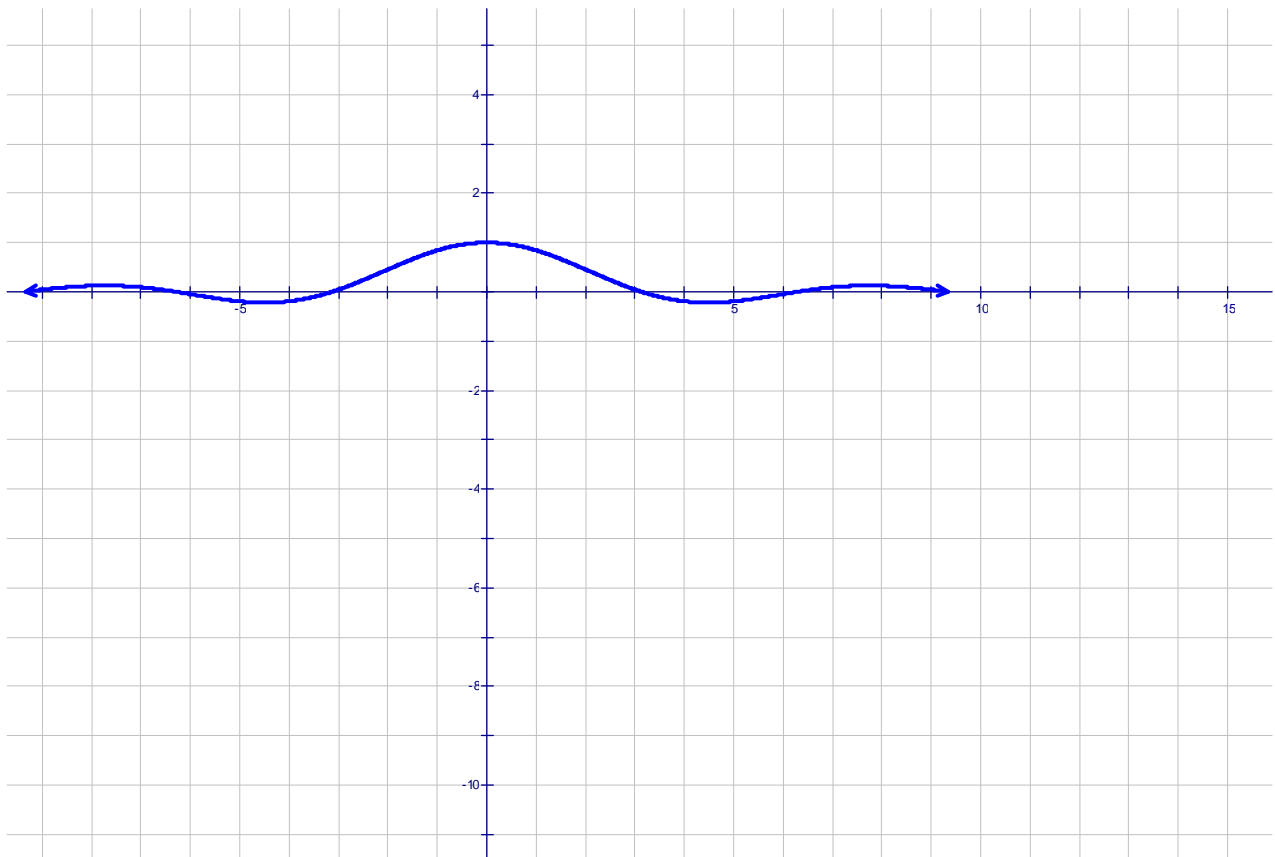
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### Example 1

$$\lim_{x \rightarrow 0} \frac{\sin x}{x}$$

$x$	$-.1$	$-.01$	$-.001$	$1.1$	$1.01$	$1.001$
$f(x)$	$f(-.1)$ $= \frac{\sin(-.1)}{-.1}$ $= .988$	$f(-.01)$ $= \frac{\sin(-.01)}{-.01}$ $= .999$	$f(-.001)$ $= \frac{\sin(-.001)}{-.001}$ $= .9999$	$f(0.1)$ $= \frac{\sin(0.1)}{0.1}$ $= .988$	$f(0.01)$ $= \frac{\sin(0.01)}{0.01}$ $= .999$	$f(0.001)$ $= \frac{\sin(0.001)}{0.001}$ $= .9999$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

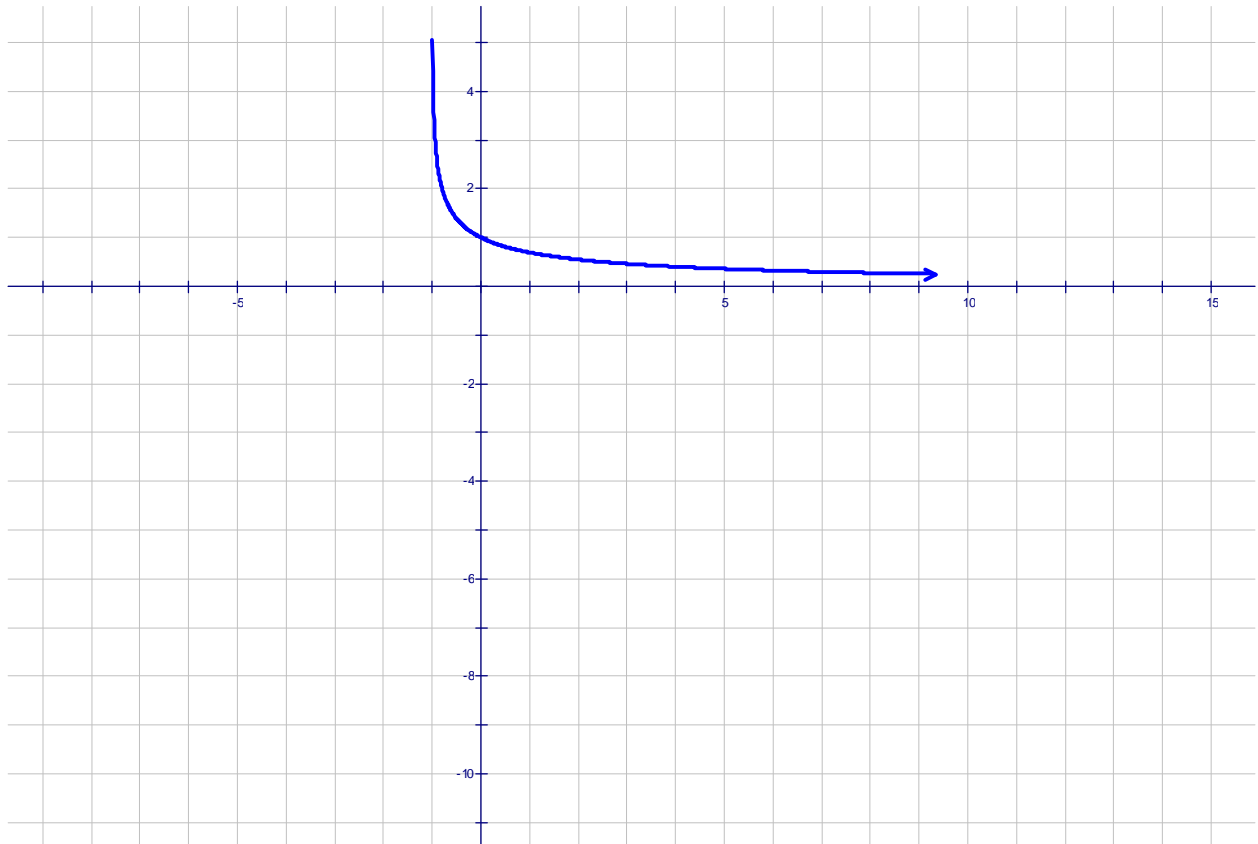


## Example 2

$$\lim_{x \rightarrow 0} \frac{\ln(x+1)}{x}$$

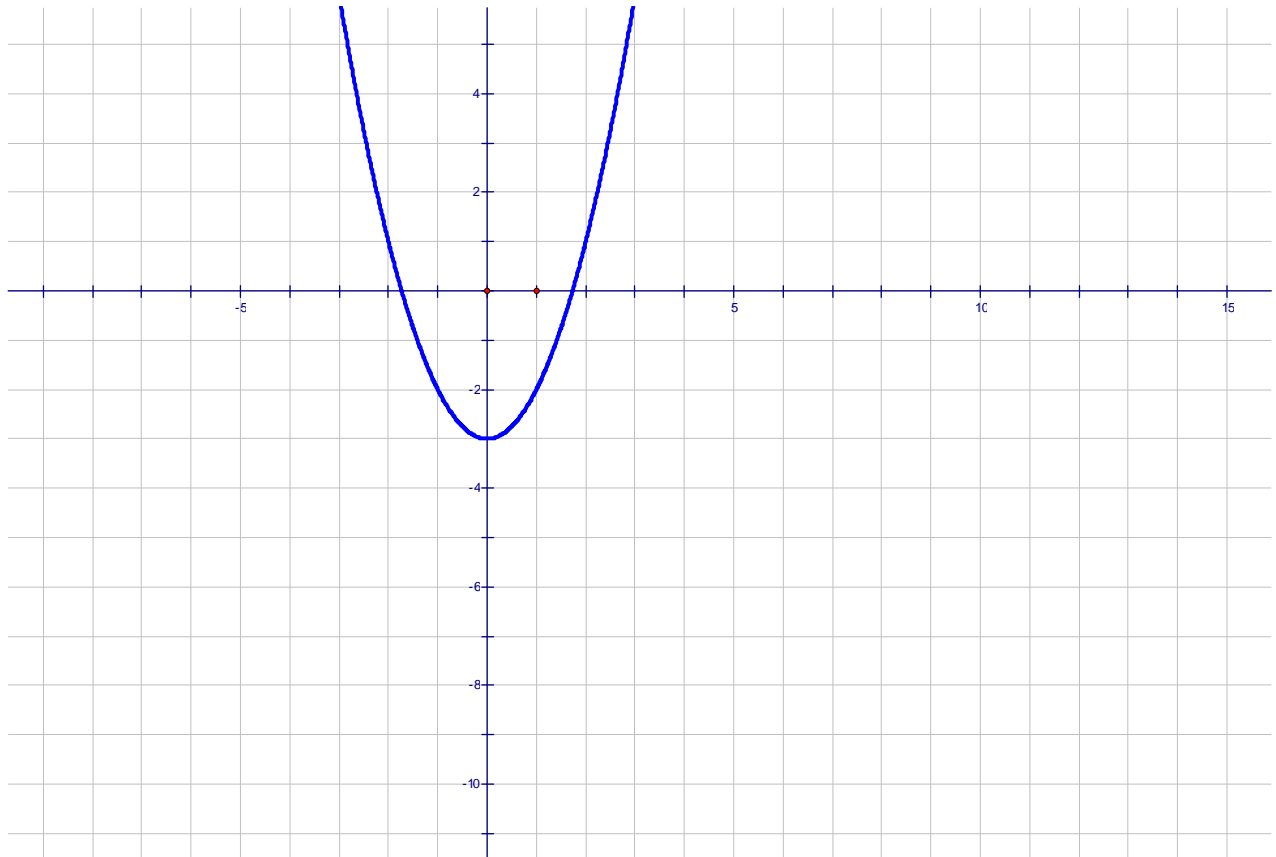
$x$	.1	.01	.001	-.1	-.01	-.001
$f(x)$	$f(.1)$ $= \frac{\ln(.1+1)}{.1}$ $= .953$	$f(.01)$ $= \frac{\ln(.01+1)}{.01}$ $= .995$	$f(.001)$ $= \frac{\ln(.001+1)}{.001}$ $= .9995$	$f(-.1)$ $= \frac{\ln(-.1+1)}{-.1}$ $= 1.05$	$f(-.01)$ $= \frac{\ln(-.01+1)}{-.01}$ $= 1.005$	$f(-.001)$ $= \frac{\ln(-.001+1)}{-.001}$ $= 1.0005$

$$\lim_{x \rightarrow 0} \frac{\ln(x+1)}{x} = 1$$



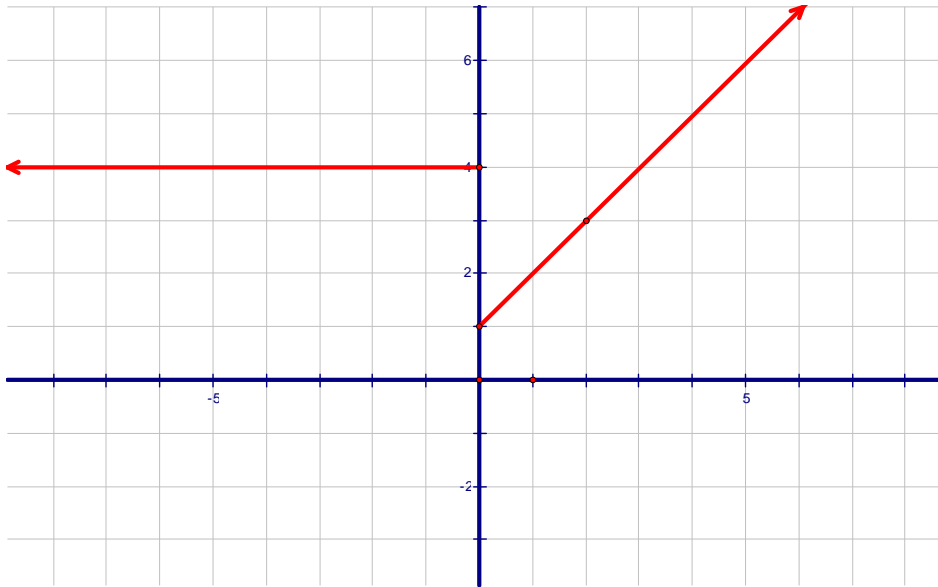
### Example 3

$$\lim_{x \rightarrow 1} x^2 - 3$$



$$\lim_{x \rightarrow 1} x^2 - 3 = 2$$

### Example 4



a) Find  $\lim_{x \rightarrow 2} f(x)$

**Solution:**  $\lim_{x \rightarrow 2} f(x) = 2$

b) Find  $\lim_{x \rightarrow -3} f(x)$

**Solution:**  $\lim_{x \rightarrow -3} f(x) = 1$

c) Find  $\lim_{x \rightarrow 0^+} f(x)$

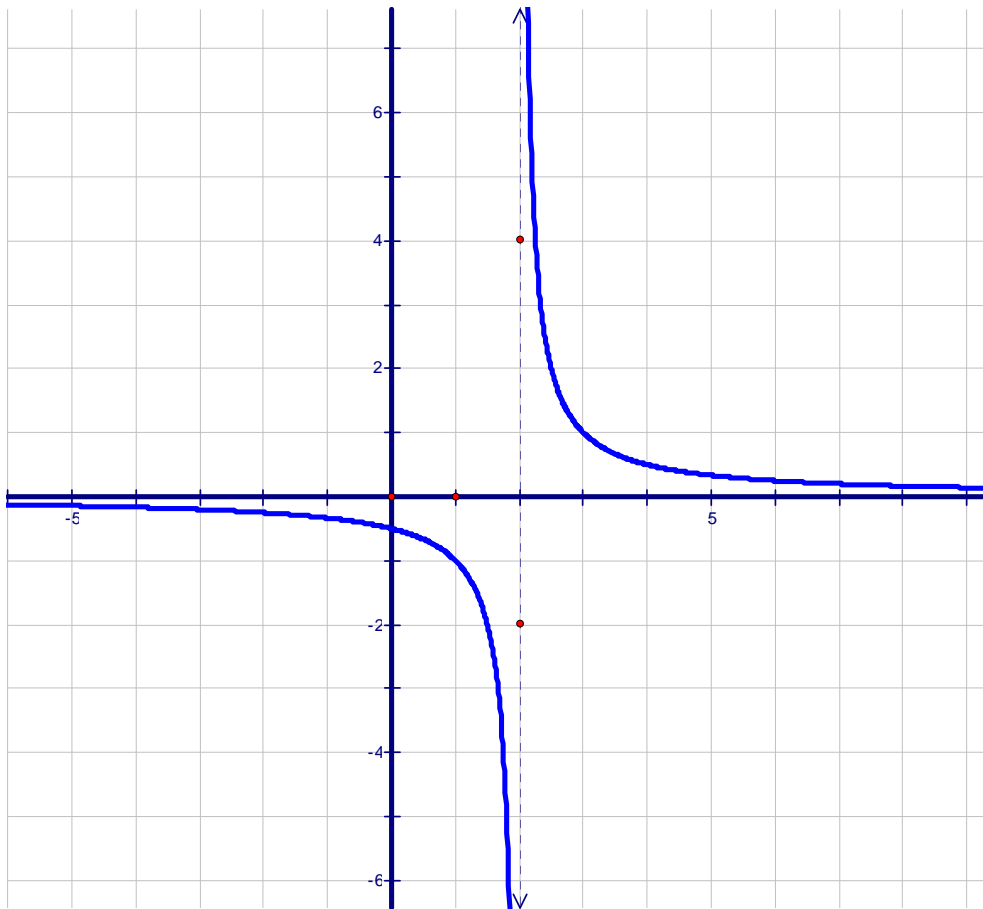
**Solution**  $\lim_{x \rightarrow 0^+} f(x) = 1$  **Left hand limit (The y value x approaches from the left)**

d) Find  $\lim_{x \rightarrow 0^-} f(x)$

**Solution:**  $\lim_{x \rightarrow 0^-} f(x) = 4$  **Right hand limit (The y value x approaches from the right)**

e) Find  $\lim_{x \rightarrow 0} f(x)$ : *Does Not Exist* **The limit doesn't exist since the right hand and the left hand limits are different.**

### Example 5



a) Find  $\lim_{x \rightarrow 2} f(x)$ :

**Solution:** No Limit (Limit does not exist)

b) Find  $\lim_{x \rightarrow 0} f(x)$ :

**Solution:**  $\lim_{x \rightarrow 0} f(x) = \frac{1}{2}$

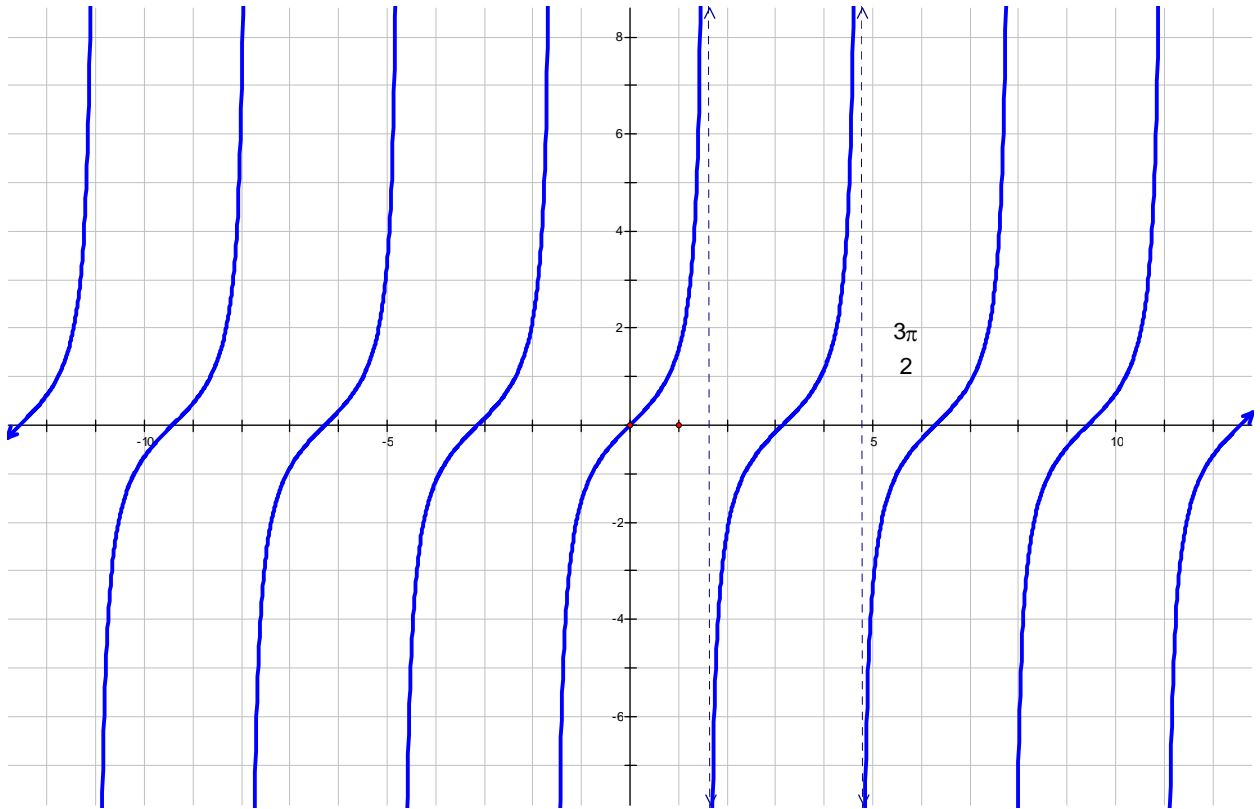
c) Find  $\lim_{x \rightarrow 3} f(x)$

**Solution:**  $\lim_{x \rightarrow 3} f(x) = 1$

### Example 6

Find the limit using the graph of the function

$$\lim_{x \rightarrow \frac{3\pi}{2}} \tan x$$



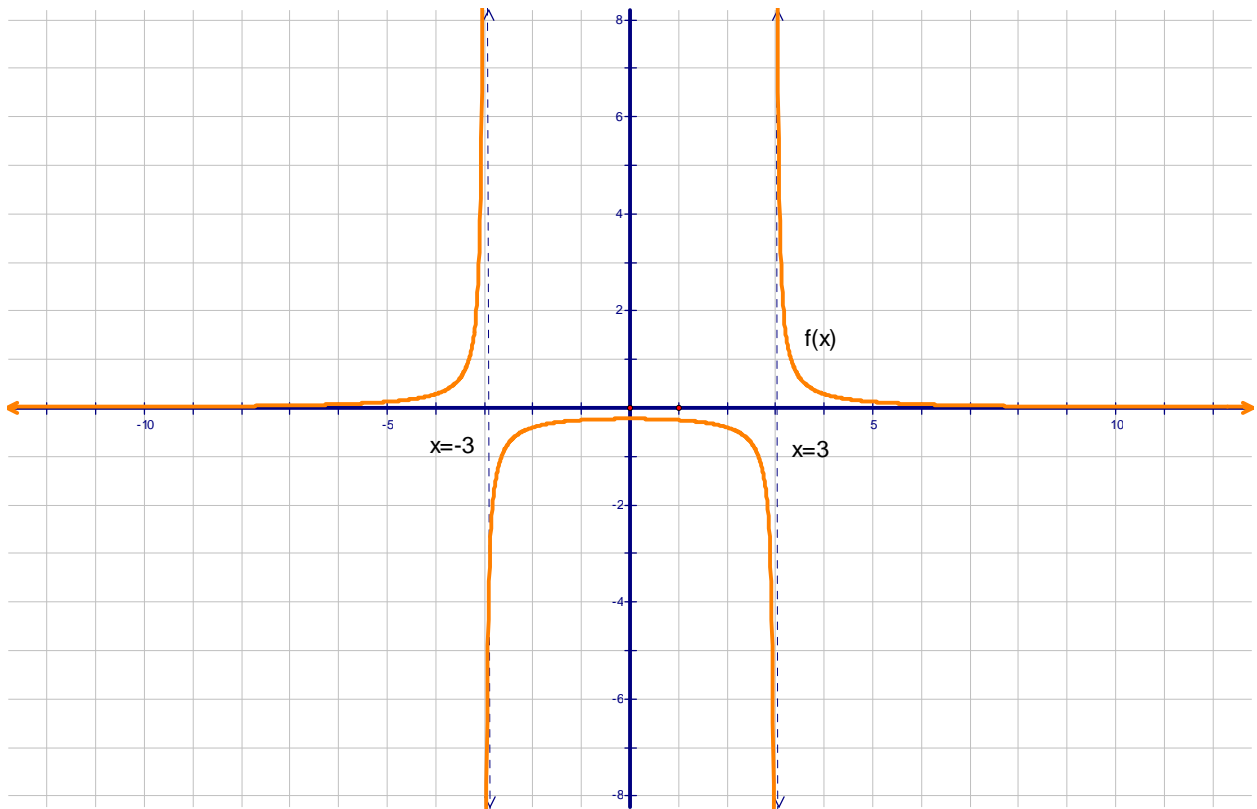
$$\lim_{x \rightarrow \frac{3\pi}{2}} \tan x \text{ (Does not exist)}$$

### Example 7

Find each limit.

a)  $\lim_{x \rightarrow -3} f(x) = \underline{\hspace{2cm}}$

b)  $\lim_{x \rightarrow 3} f(x) = \underline{\hspace{2cm}}$



**Solution:**

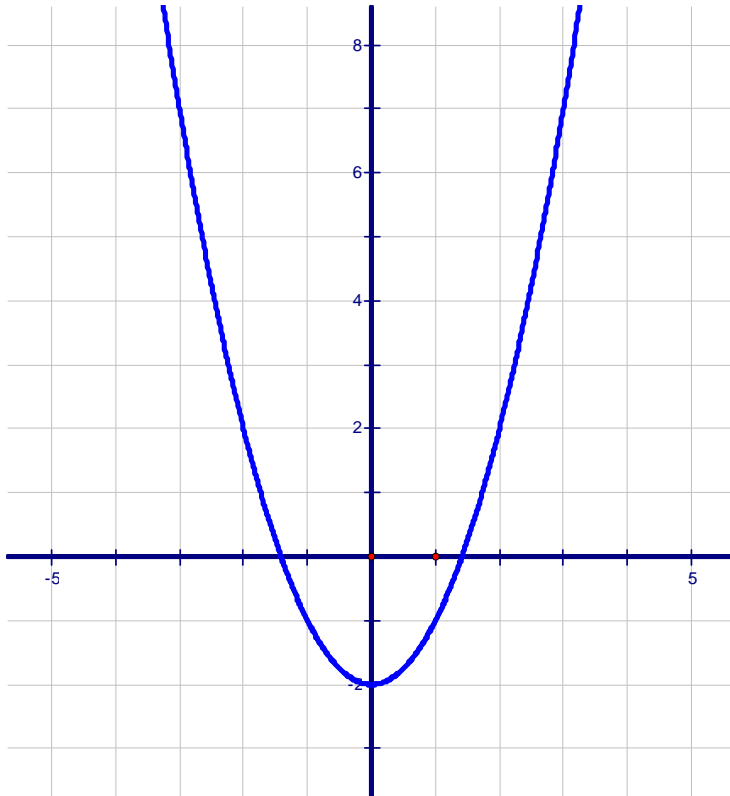
$$\lim_{x \rightarrow -3} f(x) \text{ Does not exist}$$

$$\lim_{x \rightarrow 3} f(x) \text{ Does not exist}$$

### Example 7

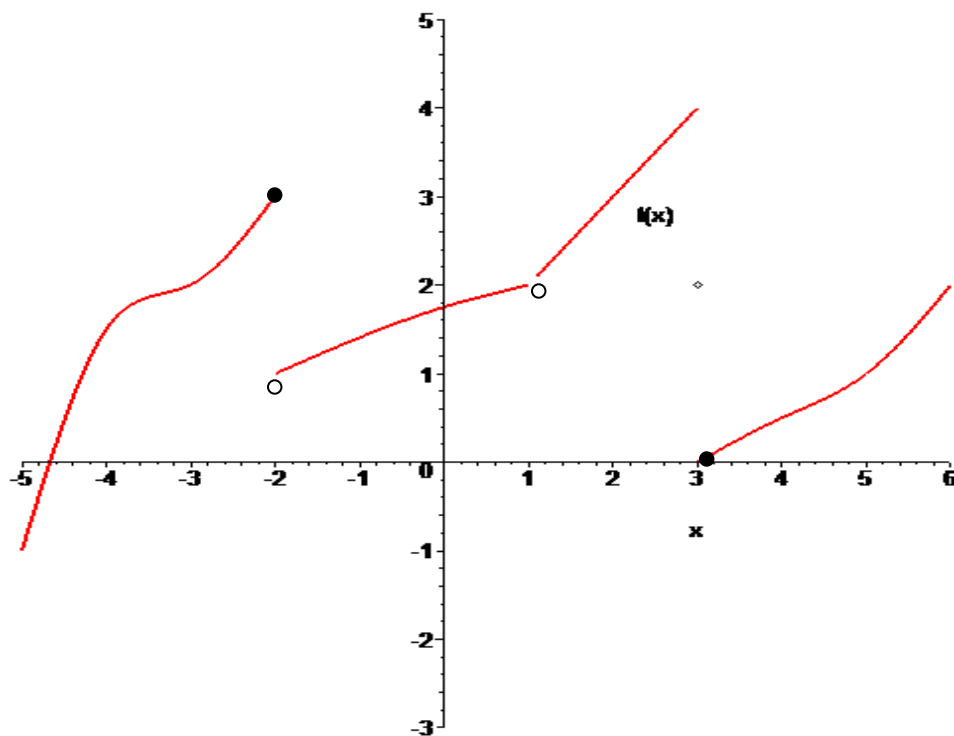
Using the graph below evaluate each limit.

- a) Find  $\lim_{x \rightarrow 2} f(x)$  : **Solution**  $\lim_{x \rightarrow 2} f(x) = 2$   
b) Find  $\lim_{x \rightarrow 0} f(x)$  : **Solution**  $\lim_{x \rightarrow 0} f(x) = -2$   
c) Find  $\lim_{x \rightarrow -2} f(x)$  : **Solution**  $\lim_{x \rightarrow -2} f(x) = 2$



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### Example 8



- a) Find  $\lim_{x \rightarrow -2^-} f(x)$ : **Solution**  $\lim_{x \rightarrow -2^-} f(x) = 3$
- b) Find  $\lim_{x \rightarrow -2^+} f(x)$ : **Solution**  $\lim_{x \rightarrow -2^+} f(x) = 1$
- c) Find  $\lim_{x \rightarrow -2} f(x)$ : **Solution**  $\lim_{x \rightarrow -2} f(x) DNE$
- d) Find  $\lim_{x \rightarrow 1^-} f(x)$ : **Solution**  $\lim_{x \rightarrow 1^-} f(x) = 2$
- e) Find  $\lim_{x \rightarrow 1^+} f(x)$ : **Solution**  $\lim_{x \rightarrow 1^+} f(x) = 2$
- f) Find  $\lim_{x \rightarrow 3^-} f(x)$ : **Solution**  $\lim_{x \rightarrow 3^-} f(x) = 2$
- g) Find  $\lim_{x \rightarrow 3^+} f(x)$ : **Solution**  $\lim_{x \rightarrow 3^+} f(x) = 0$
- h) Find  $\lim_{x \rightarrow 3} f(x)$ : **Solution**  $\lim_{x \rightarrow 3} f(x) DNE$
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### Example 9

Using the graph below evaluate each limit.

a) Find  $\lim_{x \rightarrow 5} f(x)$ : **Solution**  $\lim_{x \rightarrow 5} f(x) = \text{Does Not Exist}$

b) Find  $\lim_{x \rightarrow -7} f(x)$ : **Solution**  $\lim_{x \rightarrow -7} f(x) = -4$

c) Find  $\lim_{x \rightarrow 0} f(x)$ : **Solution**  $\lim_{x \rightarrow 0} f(x) = 2$

