

Math 151
Section 1.4
Exponential and Logarithmic Functions

Exponential Functions

Properties of Exponents

Rule	Example
1) $a^0 = 1$	$3^0 = 1$
2) $a^n a^m = a^{n+m}$	$a^4 a^5 = a^{4+5} = a^9$
3) $(a^n)^m = a^{nm}$	$(x^4)^3 = x^{4 \cdot 3} = x^{12}$
4) $(ab)^n = a^n b^n$	$(5x)^3 = 5^3 x^3 = 125x^3$
5) $\frac{a^n}{a^m} = a^{n-m}$	$\frac{a^9}{a^2} = a^{9-2} = a^7$
6) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{x}{y}\right)^4 = \frac{x^4}{y^4}$
7) $a^{-n} = \frac{1}{a^n}$	$x^{-5} = \frac{1}{x^5}$
8) $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$	$8^{\frac{4}{3}} = (\sqrt[3]{8})^4 = 2^4 = 16$

Examples of Exponential Functions

In an exponential function, the x value or domain is in the exponent of the functions. Here are a few examples of exponential functions.

$$f(x) = 2^x$$

$$g(x) = 2^{-x}$$

Use these functions evaluate the following function at the assigned value for x.

Example 1

Find $f(2)$, $f(-2)$, and $f(0)$

$$f(2) = 2^2 = 4$$

$$f(-2) = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$$

$$f(0) = 2^0 = 1$$

Example 2

Find $g(2)$, $g\left(\frac{1}{2}\right)$, and $g(0)$

$$g(2) = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$$

$$g\left(\frac{1}{2}\right) = 2^{-\frac{1}{2}} = \frac{1}{2^{\frac{1}{2}}} = \frac{1}{\sqrt{2}}$$

The Exponential Function

The **exponential function** or **Euler number** has the following approximation:

$$e \approx 2.7182818$$

Given the function $f(x) = e^x$ that contains the exponential function, evaluate these function values: $f(3)$, $f(-2)$, and $f(0)$

$$f(3) = e^3 \approx 20.086$$

$$f(-2) = e^{-2} = \frac{1}{e^2} \approx .14$$

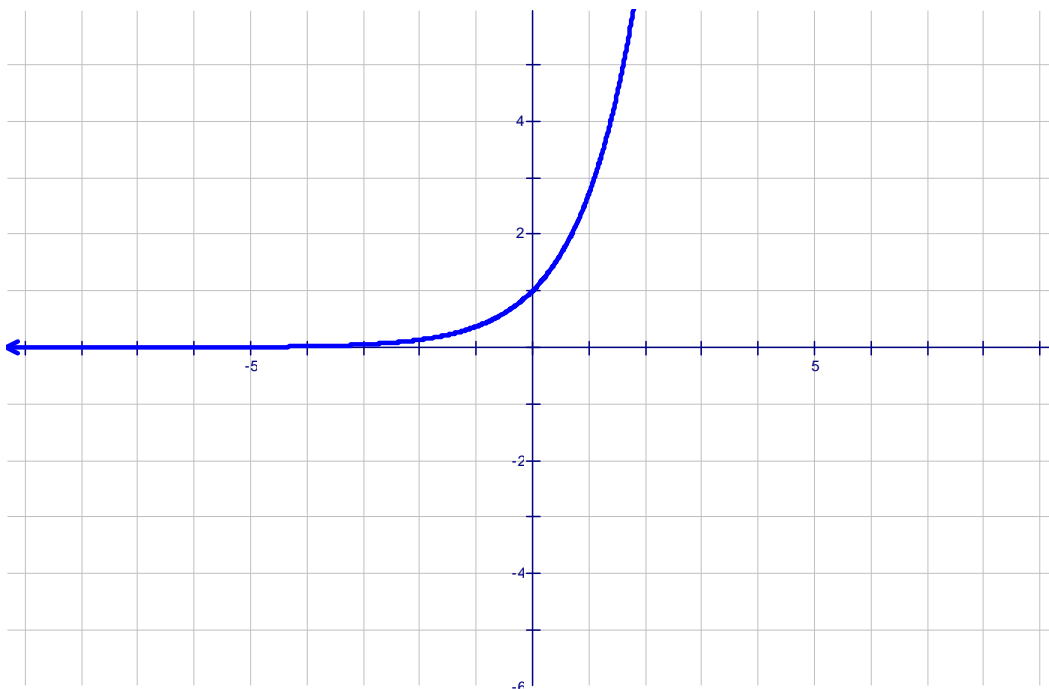
$$f(0) = e^0 = 1$$

The graph of $f(x) = e^x$

A graph of e^x can be found by making a chart. The chart contains 5 x-values and their corresponding y-values.

x	$f(x)$
-2	$f(-2) = e^{-2} = .14$
-1	$f(-1) = e^{-1} = .37$
0	$f(0) = e^0 = 1$
1	$f(1) = e^1 = 2.72$
2	$f(2) = e^2 = 7.39$

Now, using the coordinates in the above a sketch of $f(x) = e^x$ can be made.

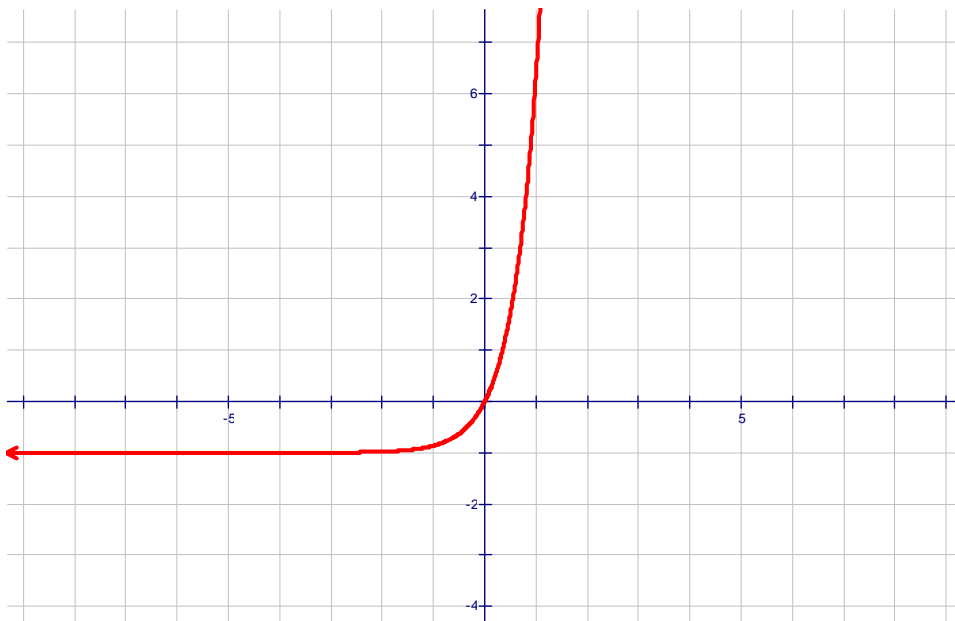


Example 3

Graph $y = e^{2x} - 1$

Make chart using the same values as the previous example and use the values to sketch a graph of $y = e^{2x} - 1$

x	$f(x)$
-2	$f(-2) = e^{2(-2)} - 1 = -.98$
-1	$f(-1) = e^{2(-1)} - 1 = -.86$
0	$f(0) = e^{2(0)} - 1 = 0$
1	$f(1) = e^{2(1)} - 1 = 6.39$
2	$f(2) = e^{2(2)} - 1 = 53.6$



Solving Exponential Equations

Example 4 (Solve each exponential function)

1) Solve $2^x = 64$

Solution:

$$\begin{aligned}2^x &= 64 \\ \Rightarrow 2^x &= 2^6 \\ \Rightarrow x &= 6\end{aligned}$$

2) Solve $2^{2x} = \frac{1}{16}$

Solution:

$$\begin{aligned}2^{2x} &= \frac{1}{16} \\ \Rightarrow 2^{2x} &= \frac{1}{2^4} \\ \Rightarrow 2^{2x} &= 2^{-4} \\ \Rightarrow 2x &= -4 \\ \Rightarrow x &= -2\end{aligned}$$

3) Solve $e^{2x-4} = e^4$

Solution:

$$e^{2x-4} = e^4 \Rightarrow 2x - 4 = 4 \Rightarrow 2x = 4 + 4 \Rightarrow 2x = 8 \Rightarrow x = 4$$

Natural Logarithms and Logarithms

Definition of a Logarithm

$$b^x = a \Leftrightarrow \log_b a = x$$

Definition of a Natural Logarithm

$$e^x = a \Leftrightarrow \ln a = x$$

The natural logarithm is a special case of a logarithm where the base is the exponential function. e

The Inverse of the Natural Logarithm

Rule 1: The natural logarithm and the exponential function are inverses of each other.

$$1) \ln e^x = x$$

$$2) e^{\ln x} = x$$

Example 5

Find the inverse of $f(x) = \ln(x + 3)$

$$\text{Let } y = \ln(x + 3)$$

To find the inverse of $y = \ln(x + 3)$, invert the x and y values and solve for y :

$$x = \ln(y + 3)$$

To solve for y , we must evaluate each side of the equation using the exponential function.

$$e^x = e^{\ln(y+3)}$$

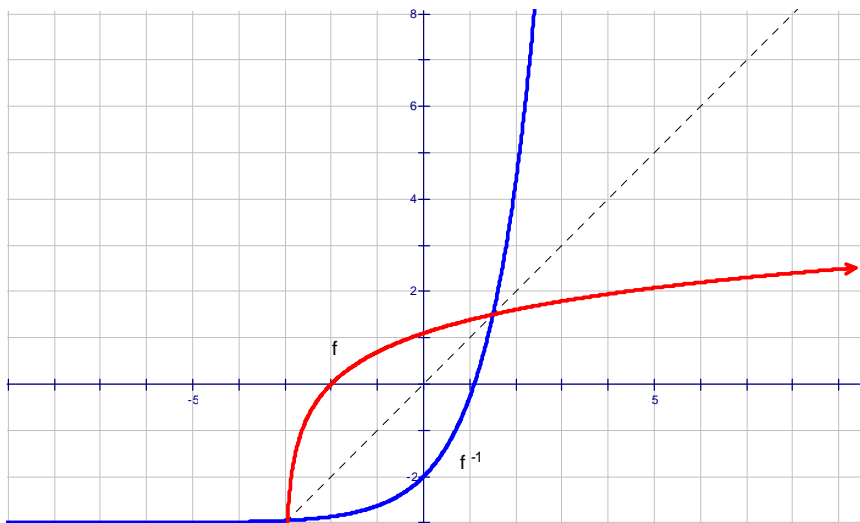
$$\Rightarrow e^x = y + 3$$

$$\Rightarrow e^x - 3 = y + 3 - 3$$

$$\Rightarrow e^x - 3 = y$$

$$\Rightarrow f^{-1}(x) = e^x - 3$$

Note: If you graph f and f^{-1} they are symmetric about the line $y = x$



Properties of the natural logarithm

$$1) \ln(xy) = \ln x + \ln y$$

$$2) \ln\left(\frac{x}{y}\right) = \ln x - \ln y$$

$$3) \ln x^y = y \ln x$$

Example 6

Use the properties of logarithms to expand expression.

$$1) \ln az$$

Solution: $\ln az = \ln a + \ln z$

$$2) \ln\left(\frac{ab}{c}\right)$$

Solution: $\ln\left(\frac{ab}{c}\right) = \ln(ab) - \ln c = \ln a + \ln b - \ln c$

$$3) \ln\left(\frac{x^2 - 1}{5x}\right)^2$$

Solution: $\ln\left(\frac{x^2 - 1}{5x}\right)^2 = 2 \ln\left(\frac{x^2 - 1}{5x}\right) = 2[\ln(x^2 - 1) - \ln 5x]$

Example 7

Use the logarithmic properties to write each logarithm as a single logarithmic expression.

$$1) \ln(x^2 + 1) + \ln(4x - 1)$$

Solution: $\ln(x^2 + 1) + \ln(4x - 1) = \ln[(x^2 + 1)(4x - 1)]$

$$2) \ln a + \ln c - 2 \ln b$$

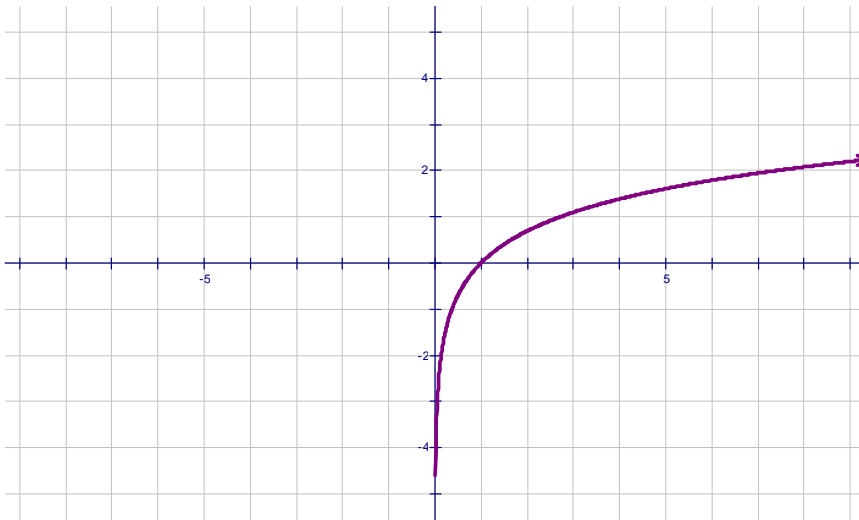
Solution: $\ln a + \ln c - 2 \ln b = \ln a + \ln c - \ln b^2 = \ln ac - \ln b^2 = \ln\left(\frac{ac}{b^2}\right)$

The graph of the natural logarithm

Graph $f(x) = \ln x$

In this case we must use x-values that are greater than zero since the domain of $y = \ln x$ is $\{x : x > 0\}$

x	$f(x) = \ln x$
1	$f(1) = \ln 1 = 0$
10	$f(10) = \ln 10 = 1$
50	$f(50) = \ln 50 = 3.9$
100	$f(100) = \ln 100 = 2$



Example 8

Use the inverse properties of e^x and $\ln(x)$ to simply each expression.

1) $e^{\ln x^3}$

Solution: $e^{\ln x^3} = x^3$

2) $\ln(e^{\sqrt{x-3}})$

Solution: $\ln(e^{\sqrt{x-3}}) = \sqrt{x-3}$

Example 10

Use the properties of the natural logarithm to solve the following equation

$$4^x = 70$$

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$$\ln(4^x) = \ln(70)$$

$$x \ln(4) = \ln(70)$$

$$x = \frac{\ln(70)}{\ln(4)}$$

$$x \approx 3.06$$

Example 11

Use the properties of the natural logarithm to solve the following equation

$$e^{x-2} = 23$$

$$e^{x-2} = 23$$

$$\ln(e^{x-2}) = \ln(23)$$

$$x - 2 = \ln(23)$$

$$x = \ln(23) + 2$$

$$x \approx 5.14$$