

Math 151
Section 1.3

Inverse Functions

Definition of an inverse function

A function $f^{-1}(x)$ is an inverse function of $f(x)$ if the following conditions are true.

$$f(f^{-1}(x)) = x \text{ and } f^{-1}(f(x)) = x$$

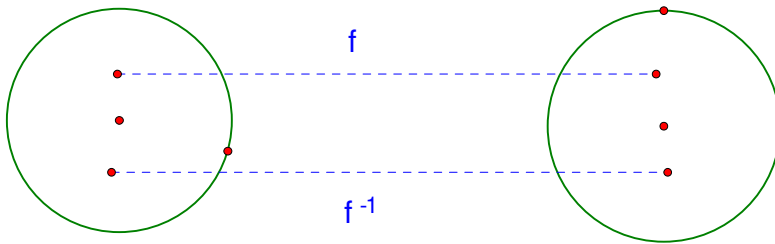
The **inverse function** is formed by interchanging the first and second coordinates.

Given the following function, find the inverse function.

$$f(x) = \{(1,2), (3,4), (4,2), (5,6)\}$$

Inverse Functions:

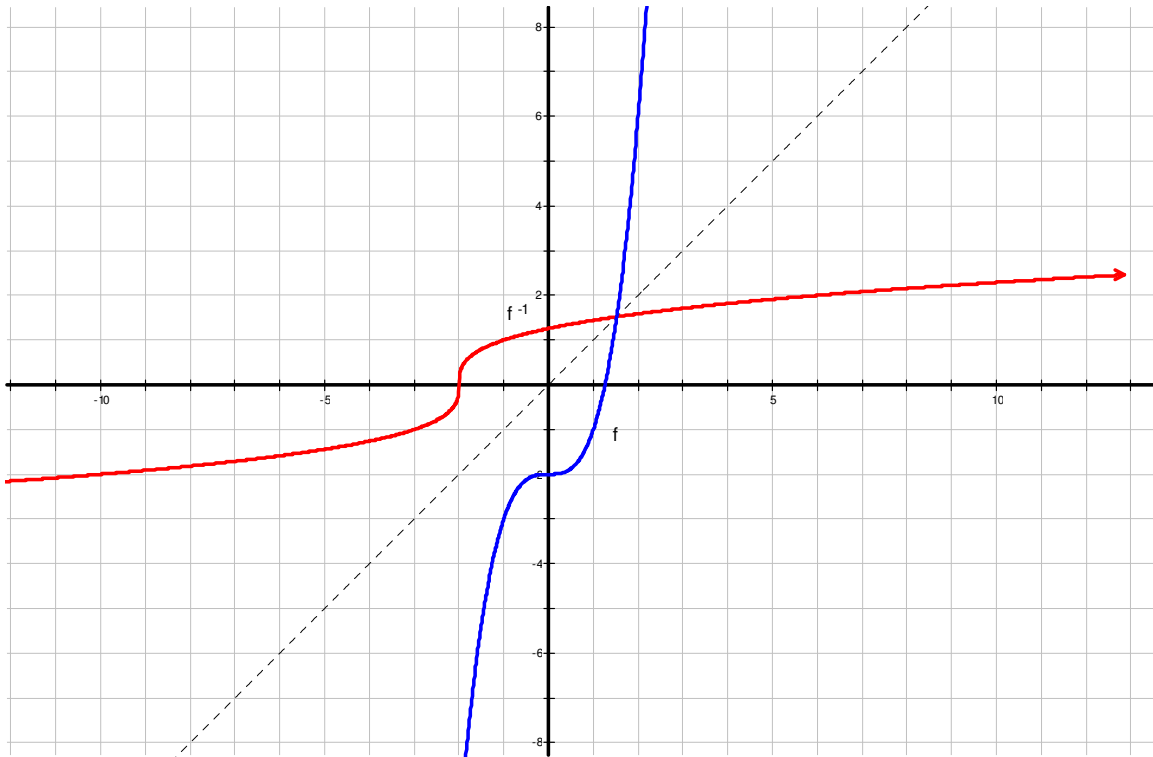
$$f^{-1}(x) = \{(2,1), (4,3), (2,4), (6,5)\}$$



Reflexive Property of Inverse Function

The graph of f contains the point (a,b) if and only if the graph of f^{-1} contains the point (b,a)

Graph of the function versus its inverse function.



The graph of a function and its inverse are symmetric about the line $y = x$.

Example 1

Show that f and g are inverses by showing that $f(g(x)) = x$ and $g(f(x)) = x$

$$f(x) = 2x + 4 : g(x) = \frac{x - 4}{2}$$

Part 1: $f(g(x)) = f\left(\frac{x - 4}{2}\right) = 2\left(\frac{x - 4}{2}\right) + 4 = (x - 4) + 4 = x$

Part 2: $g(f(x)) = g(2x + 4) = \frac{(2x + 4) - 4}{2} = \frac{2x}{2} = x$

Example 2

Show that f and g are inverses by showing that $f(g(x)) = x$ and $g(f(x)) = x$

$$f(x) = x^3 + 1 : g(x) = \sqrt[3]{x-1}$$

Part 1: $f(g(x)) = f(\sqrt[3]{x-1}) = (\sqrt[3]{x-1})^3 + 1 = x - 1 + 1 = x$

Part 2: $g(f(x)) = g(x^3 + 1) = \sqrt[3]{(x^3 + 1) - 1} = \sqrt[3]{x^3} = x$

Example 3

Show that f and g are inverses by showing that $f(g(x)) = x$ and $g(f(x)) = x$

$$f(x) = x^2 - 2 : g(x) = \sqrt{x+2} : x \geq 0$$

Part 1: $f(g(x)) = f(\sqrt{x+2}) = (\sqrt{x+2})^2 - 2 = x + 2 - 2 = x$

Part 2: $g(f(x)) = g(x^2 - 2) = \sqrt{(x^2 - 2) + 2} = \sqrt{x^2} = x$

Example 4

Find the inverse of $f(x) = x^3 - 4$

First, replace $f(x)$ with y

$$y = x^3 - 4$$

Next, invert x and y

$$x = y^3 - 4$$

Solve for y

$$x = y^3 - 4$$

$$x + 4 = y^3 - 4 + 4$$

$$x + 4 = y^3$$

$$\sqrt[3]{x + 4} = \sqrt[3]{y^3}$$

$$\sqrt[3]{x + 4} = y$$

$$\Rightarrow f^{-1}(x) = \sqrt[3]{x + 4}$$

Example 5

Find the inverse of $f(x) = 4x + 3$

First, replace $f(x)$ with y

$$y = 4x + 3$$

Next, invert x and y

$$x = 4y + 3$$

Solve for y

$$x = 4y + 3$$

$$x - 3 = 4y + 3 - 3$$

$$x - 3 = 4y$$

$$\frac{x - 3}{4} = \frac{4y}{4}$$

$$\frac{x - 3}{4} = y \Rightarrow f^{-1}(x) = \frac{x - 3}{4}$$

The existence of an inverse function

A function has an inverse if and only if it is a one-to-one function.

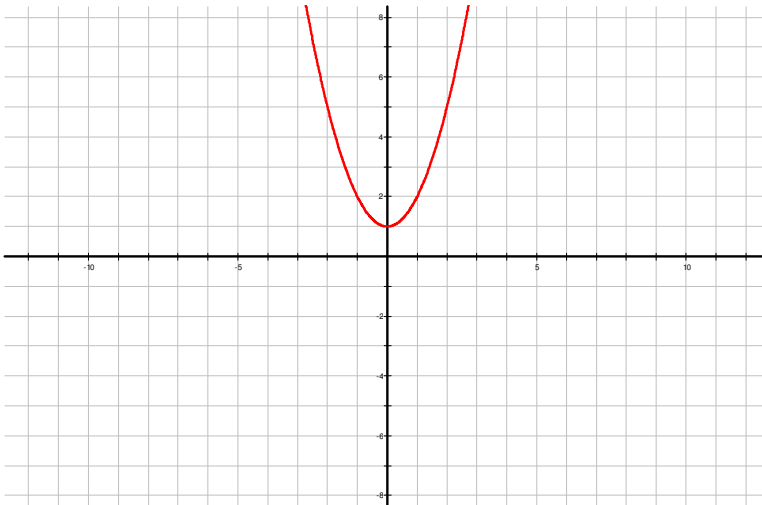
Horizontal Line Test

If a horizontal line can be drawn on a graph of a function so that it intercepts a given function more than once, the function is not one-to-one.

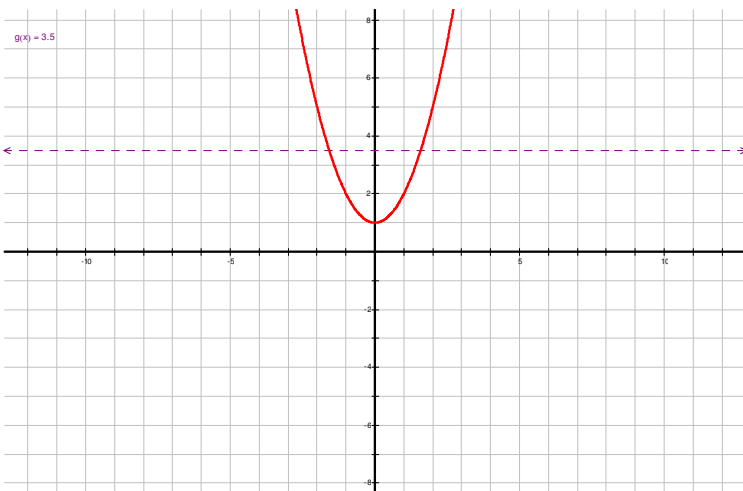
Example 6

Use the horizontal line test to determine if the function is one-to-one.

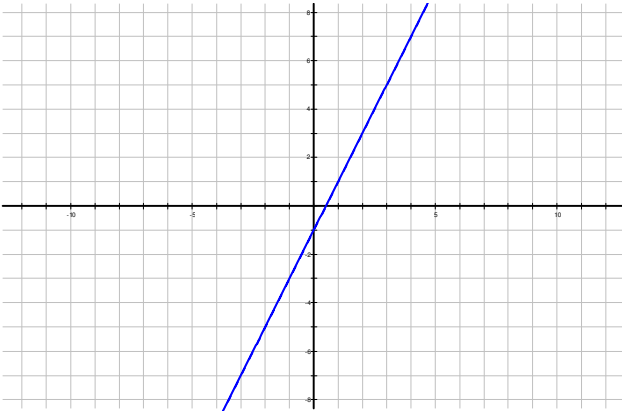
1) $f(x) = x^2 + 1$



Solution: The function fails the horizontal line test, so the function is not a one-to-one function.

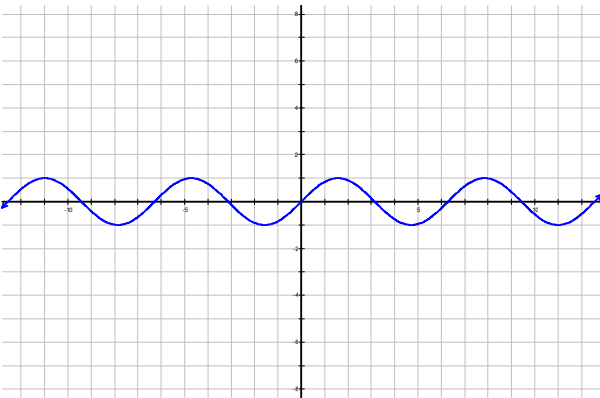


2) $f(x) = 2x - 1$

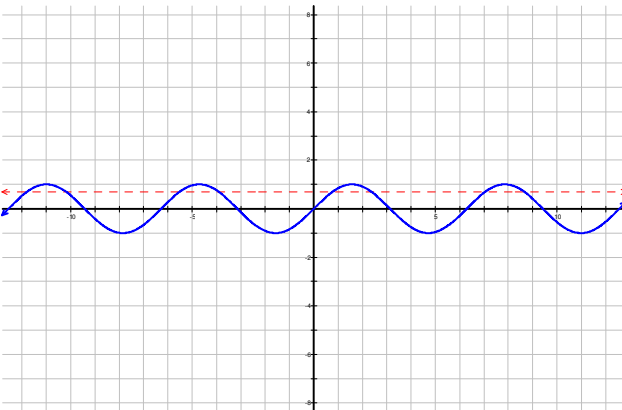


The function passes the horizontal line test, because every horizontal line intersects the function only once. Therefore, the function is one-to-one and has an inverse.

3) $f(x) = \cos(x)$



Solution: The function fails the horizontal line test, so the function is not a one-to-one function.



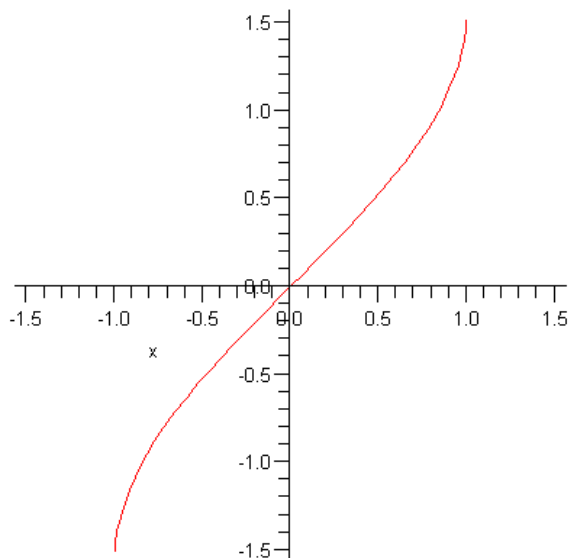
Inverse Trigonometric Functions

Function	Domain	Range
$y = \arcsin x$ iff $\sin y = x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = \arccos x$ iff $\cos y = x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$y = \arctan x$ iff $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \operatorname{arccot} x$ iff $\cot y = x$	$-\infty < x < \infty$	$0 < y < \pi$
$y = \operatorname{arcsec} x$ iff $\sec y = x$	$ x \geq 1$	$0 \leq y \leq \pi : y \neq \frac{\pi}{2}$
$y = \operatorname{arccsc} x$ iff $\csc y = x$	$ x \geq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2} : y \neq 0$

Graphs of the Inverse Functions

Example 7 (The Graph of $y = \arcsin x$)

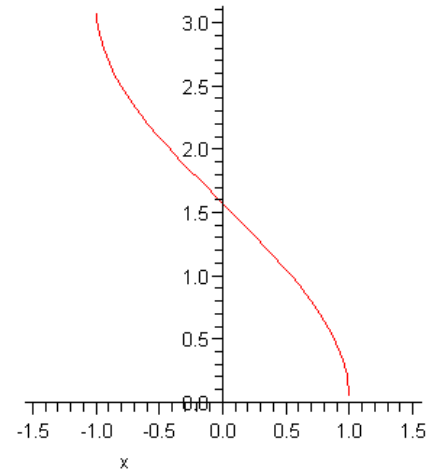
Domain: $-1 \leq x \leq 1$ or $[-1, 1]$ **Range:** $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ or $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



Example 8

The Graph of $y = \arccos x$

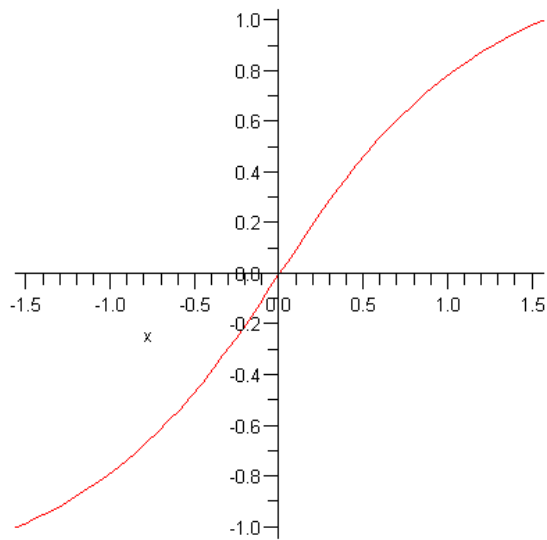
Domain: $-1 \leq x \leq 1$ or $[-1,1]$ **Range:** $0 \leq y \leq \pi$ or $[0,\pi]$



Example 9

The Graph of $y = \arctan x$

Domain: $-\infty < x < \infty$ or $(-\infty, \infty)$ **Range:** $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ or $[-\frac{\pi}{2}, \frac{\pi}{2}]$



Example 10

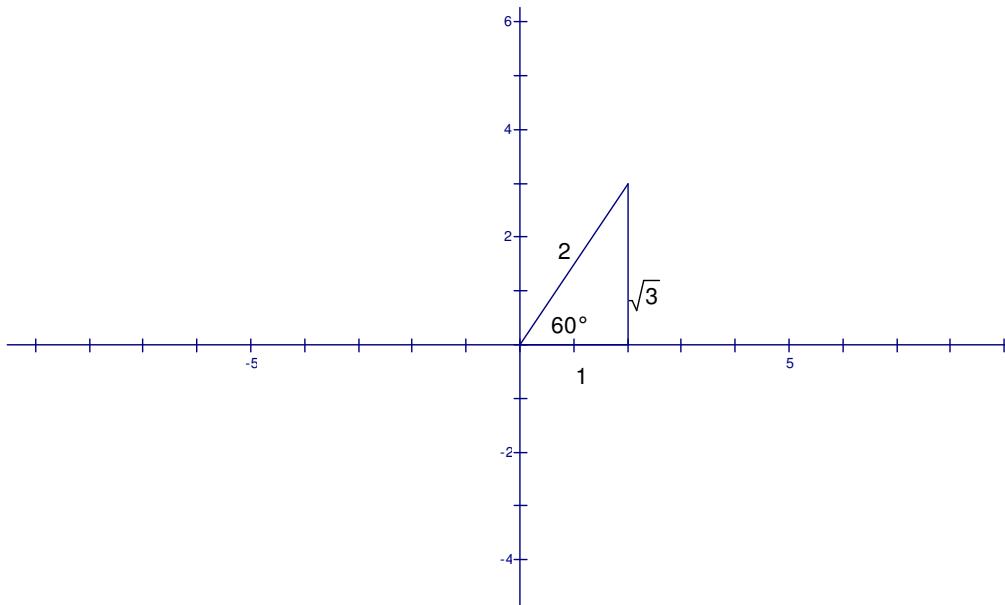
Evaluate each of the following inverse trig functions

1) Evaluate $\arcsin\left(\frac{\sqrt{3}}{2}\right)$

$$\arcsin\left(\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow \sin y = \frac{\sqrt{3}}{2} \text{ on } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\Rightarrow y = 60^\circ \text{ or } y = \frac{\pi}{3}$$

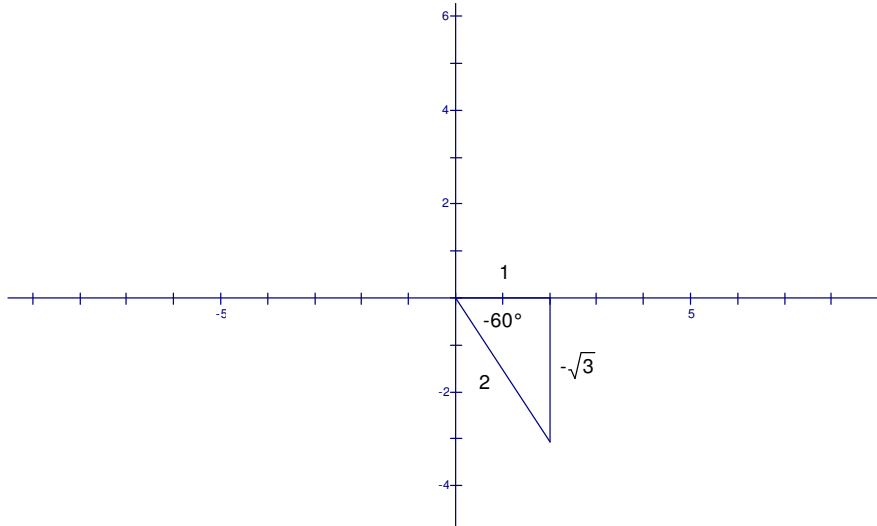


2) Evaluate $\arctan(2)$

$$\arctan(-\sqrt{3})$$

$$\Rightarrow \tan y = -\sqrt{3}$$

$$\Rightarrow y = -\frac{\pi}{3}$$



3) Evaluate $\arccos(0)$

$$\arccos(0) \Rightarrow \cos y = 0 \Rightarrow y = \frac{\pi}{2}$$

Example 11

Find the inverse of $f(x) = x^3 + 2$

First, replace $f(x)$ with y

$$y = x^3 + 2$$

Next, invert x and y

$$x = y^3 + 2$$

Solve for y

$$x = y^3 + 2$$

$$\Rightarrow x - 2 = y^3 + 2 - 2$$

$$\Rightarrow x - 2 = y^3$$

$$\Rightarrow \sqrt[3]{x-2} = \sqrt[3]{y^3}$$

$$\Rightarrow \sqrt[3]{x-2} = y$$

$$\Rightarrow f^{-1}(x) = \sqrt[3]{x-2}$$

