Euclidean verses Non Euclidean Geometries

Euclidean Geometry

Euclid of Alexandria was born around 325 BC. Most believe that he was a student of Plato. Euclid introduced the idea of an axiomatic geometry when he presented his 13 chapter book titled *The Elements of Geometry*. The Elements he introduced were simply fundamental geometric principles called axioms and postulates. The most notable are Euclid's five postulates which are stated in the next passage.

1) Any two points can determine a straight line.
2) Any finite straight line can be extended in a straight line.
3) A circle can be determined from any center and any radius.
4) All right angles are equal.
5) If two straight lines in a plane are crossed by a transversal, and sum the interior angle of the same side of the transversal is less than two right angles, then the two lines extended will intersect.

According to Euclid, the rest of geometry could be deduced from these five postulates. Euclid's fifth postulate, often referred to as the Parallel Postulate, is the basis for what are called Euclidean Geometries or geometries where parallel lines exist. There is an alternate version to Euclid fifth postulate which is usually stated as "Given a line and a point not on the line, there is one and only one line that passed through the given point that is parallel to the given line. This is a short version of the Parallel Postulate called **Fairplay's Axiom** which is named after the British math teacher who proposed to replace the axiom in all of the schools textbooks. Some individuals have tried to prove the parallel postulate, but after more than two thousand years it still remains unproven. For many centuries, these postulates have assumed to be true. However, some mathematics believed that the Euclid Fifth Postulate was suspect or incomplete. As a result, mathematicians have written alternate postulates to the Parallel Postulate. These postulates have led the way to new geometries usually called Non-Euclidean Geometries.

Non-Euclidean Geometries

In later 18th century Carl Friedrich Gauss became interested in proving Euclid Fifth Postulate as a young teenage student. After attempt at proving the postulate, young Gauss decided to try a different route. Gauss then wrote what is called Gauss's Alternate to the Parallel Postulate.

Gauss's Alternate to the Parallel Postulate

Through a given point not on a line, there are at least two lines parallel to the given line through the given point.

This alternate postulate was proposed to Russian mathematician Nikolai Lobachevsky in 1826. Lobachevsky became very interested in the problem and provided a detailed investigation into the problem of the Parallel Postulate. He concluded that Gauss's Postulate was an independent postulate and it could be changed to produce a new geometry. Ironically, Lobachevsky called the geometry an imaginary geometry because he could not comprehend the model for this type of geometry. Instead, he proposed that was possible to construct such a geometry. For this reason, this type was referred to a Lobachevskian Geometry. Today, Lobachevskian geometries are often referred to as hyperbolic geometries.

Bernard Riemann

Born in 1926, Riemann was a student Gauss further studied Gauss's work on non-Euclidean geometries. Gauss was able to develop his own alternate to the Parallel Postulate.

Reimann's Alternate to the Parallel Postulate

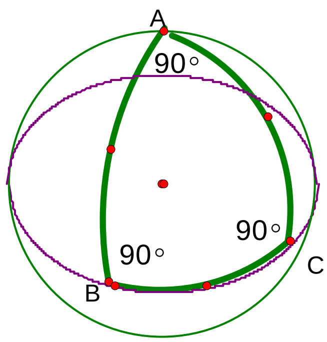
Through a given point not on a given line, there exist no lines parallel to the line through the given point.

Gauss's Alternate to the Parallel Postulate created the idea of geometries where parallel lines are non-existent. The non-Euclidean geometry developed by Gauss could be model on a sphere where as Lobachevskian's geometry had no physical model. For this reason, Riemannian geometries are also referred to as a spherical geometry or elliptical geometry. Riemann also made several contributions in areas of calculus and physics before he died of tuberculosis at age 39.

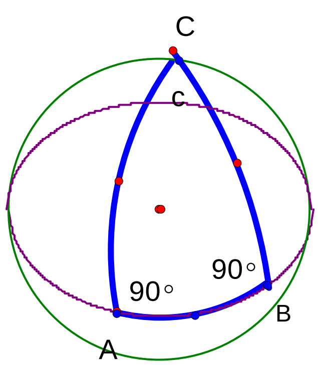
Spherical Triangles

In Riemannian geometry, geometric shapes such as triangles have a different appearance than what they would in Euclidean geometry. A spherical triangle may have sides that curvature which would allow these triangles to have some special properties. For example a spherical triangle can have more one right angle and a sum of interior angles that is more than 180 degrees.

A spherical triangle with three right angles



A spherical triangle with two right angles



The Area of Spherical Triangle

The area S of the spherical triangle ABC on sphere with radius r is given by

$$S = (m\angle A + m\angle B + m\angle C - 180^\circ) \cdot \left(\frac{\pi}{180^\circ}\right) r^2$$

where each angle is measured in degrees.

Example 1

Find the area of spherical triangle with three right angles and a radius of 2 feet. Find both the approximate and exact values for the area to the nearest hundredth.

$$m\angle A = 90^\circ$$
$$m\angle B = 90^\circ$$
$$m\angle C = 90^\circ$$
$$r = 2 \ \ feet$$
$$S = (m\angle A + m\angle B + m\angle C - 180^\circ)\cdot\left(\frac{\pi}{180^\circ}\right)r^2$$
$$S = (90^\circ + 90^\circ + 90^\circ - 180^\circ)\left(\frac{\pi}{180^\circ}\right)(2 \ ft)^2$$
$$S = (270^\circ - 180^\circ)\left(\frac{\pi}{180^\circ}\right)(2 \ \ ft)^2$$
$$S = (90^\circ)\left(\frac{\pi}{180^\circ}\right)(4 \ \ ft^2)$$
$$S = 2\pi \ ft^2 \approx 6.28 \ \ ft^2$$

Example 2

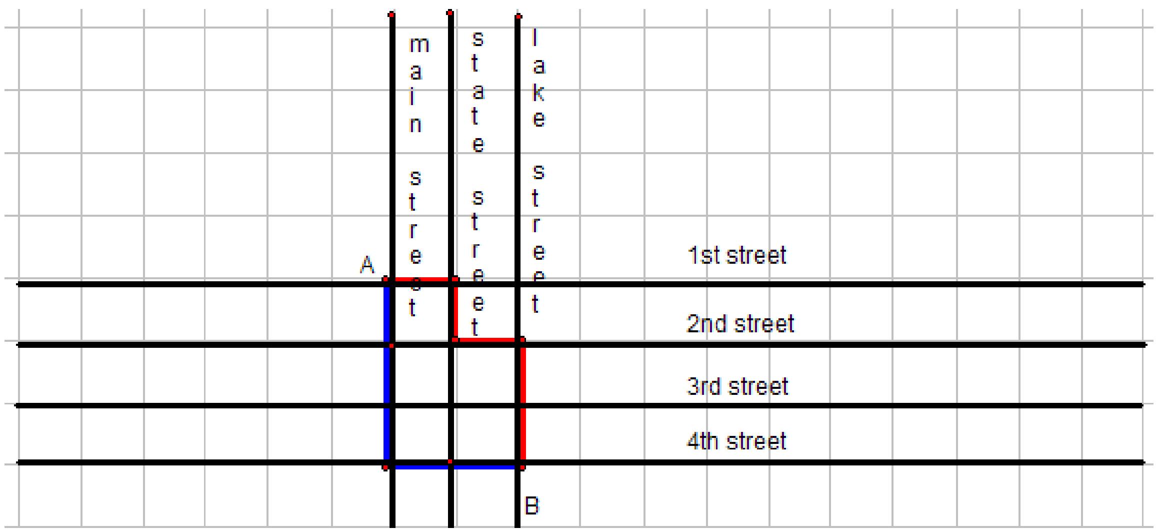
Find the area of spherical triangle with two right angles and a third angle that measures 75 degrees and a radius that measures 3 feet. Find both the approximate and exact values for the area to the nearest hundredth.

$$m\angle A = 90^\circ : m\angle B = 90^\circ : m\angle C = 75^\circ : r = 3 \ \ feet$$
$$S = (90^\circ + 90^\circ + 75^\circ - 180^\circ)\left(\frac{\pi}{180^\circ}\right)(3 \ ft)^2 = (255^\circ - 180^\circ)\left(\frac{\pi}{180^\circ}\right)(3 \ \ ft)^2 = (75^\circ)\left(\frac{\pi}{180^\circ}\right)(9 \ \ ft^2)$$
$$= \frac{135}{36}\pi \ ft^2 \approx 11.78 \ \ ft^2$$

Contemporary Non-Euclidean Geometries

The City Distance Formula

Suppose that you a city that is laid out on in grid where the street either run north and south or east and west. If you need to travel from one point in the city to another point in the city that is not on the same street, then you wound not be able to travel in a straight line. Instead, you would walk along the streets until you would reach your destination. For example, suppose you need to walk from the City Library located on the corner of First Street and Main Street to the Post Office located on the corner of Lake Street and Fourth Street. In the city map below point A represents the library and point B represents the post office.



Using the map above, two paths, the red and blue, are outlined from point A to point B or from the library to the post office. If you were to walk along either path, it turns out you would walk 5 blocks. The distance of each little path that is traveled along the city blocks can be represented by the city distance formula.

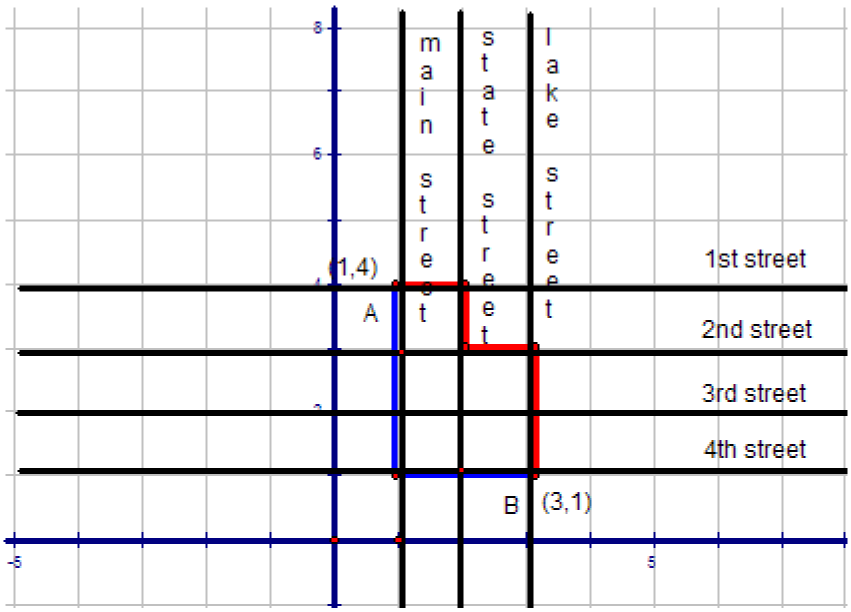
The City Distance Formula

If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points in a city, then the city distance between the point P and Q is given by the following formula.

$$d(P,Q) = |x_2 - x_1| + |y_2 - y_1|$$

Example 3

Using the city grid above with point A as the library and point B as the post office, let point A have the coordinates of (1,4) and point B have the coordinates of (3,1) Use the city distance formula to find the distance between points A and B.



Solution:

$$d(P,Q)=|x_2-x_1|+|y_2-y_1|=|3-1|+|1-4|=|2|+|3|=2+3=5 \text{ city blocks}$$

Example 4

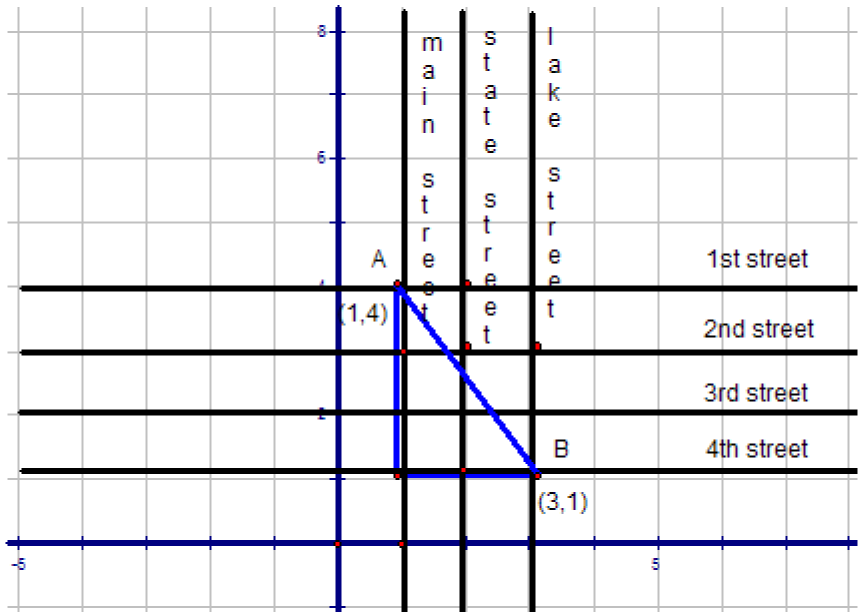
Use the city distance formula to find the distance between the points P(2,3) and Q(7,8).

Solution:

$$d(P,Q)=|x_2-x_1|+|y_2-y_1|=|8-3|+|7-2|=|5|+|5|=5+5=10 \text{ city blocks}$$

The Euclidean Distance Formula

Now let's we refer back to the example 3 with the Library, point A, located at point (1,1) and the post office, point B, located at point (3,4). What if we wanted find shortest distance between the points A and B. The easiest way to accomplish this is to make a right triangle and find the length of the hypotenuse as shown in the next diagram.



If you count the blocks in the diagram, the two legs of the triangle turn out to be 2 blocks and 3 blocks. Next, use the Pythagorean Theorem to find the hypotenuse.

$$c^2 = a^2 + b^2$$
$$c^2 = 3^2 + 2^2$$
$$c^2 = 9 + 4$$
$$c^2 = 13$$
$$\sqrt{c^2} = \sqrt{13}$$
$$c = \sqrt{13} \approx 3.6 \ blocks$$

The **Euclidean distance formula** basically find the distances between two points as shown above but use the actually coordinates instead of counting the block in the diagram.

Euclidean Distance Formula

If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points in a city, then the Euclidean distance between the point P and Q is given by the following formula.

$$d(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Now, let’s use the Euclidean distance formula to find the distance between point A(1,4) and B(3,1).

$$d(P,Q)=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(3-1)^2+(4-1)^2}=\sqrt{2^2+3^2}=\sqrt{4+9}=\sqrt{13}=3.6\ blocks$$

Using the Euclidean distance formula is essentially the same using the Pythagorean Theorem to find the distance between two points.

Example 5

Find the city distance and Euclidean distance between the points (2,3) and (10,12). (Round answers to the nearest tenth of a block)

Part 1: Find the city distance between the points (2,3) and (10,12)

$$d(P,Q)=|x_2-x_1|+|y_2-y_1|=|10-2|+|12-3|=|8|+|9|=8+9=17\ blocks$$

Part 2: Find the Euclidean distance between the points (2,3) and (10,12)

$$d(P,Q)=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(10-2)^2+(12-3)^2}=\sqrt{8^2+9^2}=\sqrt{64+81}=\sqrt{145}=12.0\ blocks$$