Section 5.2

Logarithms

Common Logarithms

$\log_b a = x \Leftrightarrow b^x = a$

$\ln a = x \Leftrightarrow e^x = a$

$\log_{10} a = x \Leftrightarrow 10^x = a$

Example 1

Write each expression as a logarithm expression

1) $3^4 = 81$

$$3^4 = 81$$
$$\log_3 81 = 4$$

2) $e^2 = 7.39$

$$e^2 = 7.39$$
$$\ln(7.39) = 2$$

3) $10^4 = 10000$

$$10^4 = 10000$$
$$\log 10000 = 4$$

Example 2

Write each expression as a exponential expression

1) $\log_2 128 = 7$

$$\log_2 128 = 7 \Rightarrow 2^7 = 128$$

2) $\ln 2.71 = 1$

$\ln 2.71 = 1 \Rightarrow e^1 = 2.71$

3) $\log 1000 = 3$

$\log 1000 = 3 \Rightarrow 10^3 = 1000$

Example 3

Solve for x by writing the equation in the exponential form.

a) $\log_4 x = 3$

Solution:

$\log_4 x = 3$

$4^3 = x$

$x = 4^3$

$x = 64$

b) $\log_5(2x-5) = 3$

Solution:

$\log_5(2x-5) = 3$

$2x-5 = 5^3$

$2x-5 = 125$

$2x-5+5 = 125+5$

$2x = 130$

$x = 65$

c) $\log_8 x = \frac{1}{3}$

Solution:

$\log_8 x = \frac{1}{3} \Rightarrow x = 8^{\frac{1}{3}} = \sqrt[3]{8} = 2$

Rule 1: $\ln e^{x} = x$

Rule 2: $e^{\ln x} = x$

Example 5

Show that the following functions are inverses of each other

$$f(x) = \ln x$$
$$g(x) = e^{x}$$

$$h(g(x)) = h(e^{x}) = \ln(e^{x}) = x$$

$$g(h(x)) = g(\ln x) = e^{\ln x} = x$$

Example 6

Show that the following functions are inverses of each other

$$f(x) = \ln x + 3$$
$$g(x) = e^{x-3}$$

$$h(g(x)) = h(e^{x-3}) = \ln\left(e^{x-3}\right) + 3 = x - 3 + 3 = x$$

$$g(h(x)) = g(\ln x - 3) = e^{\ln x - 3 + 3} = \ln e^{x} = x$$

Logarithm Laws

1) $\ln AB = \ln A + \ln B$ *or* $\log(AB) = \log A + \log B$

2) $\ln\left(\frac{A}{B}\right) = \ln A - \ln B$ *or* $\log(AB) = \log A + \log B$

3) $\ln(x^n) = n \ln x$ *or* $\log(x^n) = n \log x$

Example 7

Write each expression as a single logarithm.

1) $\ln(3x+5) - \ln(3x-5)$

$$\ln(3x+5) - \ln(3x-5) = \ln\left(\frac{3x+5}{3x-5}\right)$$

2) $3\ln x + 2\ln y - 2\ln z$

$$3\ln x + 2\ln y - 2\ln z = \ln x^3 + \ln y^2 - \ln z^2 = \ln(x^3 y^2) - \ln z^2 = \ln\left(\frac{x^3 y^2}{z^2}\right)$$

3) $2\ln x + \ln(x+4) - \ln(x+2)$

$$2\ln x + \ln(x+4) - \ln(x+2) = \ln x^2 + \ln(x+4) - \ln(x+2) = \ln(x^2(x+4)) - \ln(x+2) = \ln\left(\frac{x^2(x+4)}{x+2}\right)$$

Example 8

Solve $e^{x+4} = 6$

$$
\begin{aligned}
&e^{x+4} = 6 \\
&\ln e^{x+4} = \ln 6 \\
&(x+4)\ln e = \ln 6 \\
&x+4 = \ln 6 \\
&x = \ln 6 - 4 \approx -2.21
\end{aligned}
$$

Example 9

Solve $3^x = 12$

$$
\begin{aligned}
&3^x = 12 \\
&\ln 3^x = \ln 12 \\
&x\ln 3 = \ln 12 \\
&x = \frac{\ln 12}{\ln 3} \\
&x \approx 2.26
\end{aligned}
$$

Example 10

Solve $6^x = 88$

$$
\begin{aligned}
&6^x = 88 \\
&\ln 6^x = \ln 88 \\
&x\ln 6 = \ln 88 \\
&x = \frac{\ln 88}{\ln 2} \\
&x \approx 2.499
\end{aligned}
$$

Example 11

Solve $e^{\ln x} = 4$

$$e^{\ln x} = 4$$
$$x = 4$$

Example 12

Find the inverse of $f(x) = e^x + 4$

$$f(x) = e^x + 4$$
$$y = e^x + 4$$
$$x = e^y + 4$$
$$x - 4 = e^y + 4 - 4$$
$$\ln(x-4) = \ln(e^y)$$
$$\ln(x-4) = y$$
$$f^{-1}(x) = \ln(x-4)$$

Example 13

Find the inverse of $f(x) = \ln(x) + 3$

$$f(x) = \ln(x) + 3$$
$$y = \ln(x) + 3$$
$$x = \ln(y) + 3$$
$$x - 3 = \ln(y)$$
$$e^{x-3} = e^{\ln(y)}$$
$$e^{x-3} = y$$
$$f'(x) = e^{x-3}$$

Example 14

Find the inverse of $f(x) = e^{x+5}$

$$f(x) = e^{x+5}$$
$$y = e^{x+5}$$
$$x = e^{y+5}$$
$$\ln(x) = \ln\left(e^{y+5}\right)$$
$$\ln(x) = y + 5$$
$$\ln(x) - 5 = y + 5 - 5$$
$$\ln(x) - 5 = y$$
$$f^{-1}(x) = \ln(x) - 5$$

Example 15

Suppose $4000 is placed in a saving account for 2 years that compound interest semiannually at a rate of 2% per year. How much money would be in the saving account after 2 years?

$$P = \$4000$$
$$T = 2$$
$$R = .02$$
$$n = 2$$

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$
$$A = \$4000\left(1 + \frac{.02}{2}\right)^{2\cdot 2}$$
$$A = \$4000(1 + .01)^4$$
$$A = \$4000(1.01)^4$$
$$A = \$4000(1.0406)$$
$$A = \$4162.41$$

Example 16 (Calculating Time)

How long would it take to save $15,000, if $10,000 was placed in saving account that compounds interest quarterly at a rate of 1.0 % per year?

$P = 10{,}000$

$A = 15{,}000$

$T = ?$

$n = 4$

$R = .01$

$$A = P\left(1 + \frac{R}{n}\right)^{nt}$$

$$15{,}000 = 10{,}000\left(1 + \frac{.01}{4}\right)^{4t}$$

$$15{,}000 = 10{,}000(1 + .0025)^{4t}$$

$$15{,}000 = 10{,}000(1.0025)^{4t}$$

$$\frac{15{,}000}{10{,}000} = \frac{10{,}000(1.0025)^{4t}}{10{,}000}$$

$$1.5 = (1.0025)^{4t}$$

$$\log(1.5) = \log(1.0025)^{4t}$$

$$\log(1.5) = 4t\log(1.0025)$$

$$.17609 = 4t(.00108)$$

$$.17609 = .00432t$$

$$t = \frac{.17609}{.00432}$$

$$t \approx 40.8 \ \mathit{years}$$