Section 2.1

Set and Set Operators

Definition of a set

A **set** is a collection of objects, things or numbers.

Sets are collection of objects that can be displayed in different forms. Two of these forms are called **Roster Method** and **Builder Set Notation.**

Roster Method: In roster method, the elements of the set are listed in brackets and separated by commons. The sets in the above examples are in roster form.

$\{1,2,3,4,5\}$
$\{Ron, John, Mark, Phil\}$
$\{Virginia, West\ Virginia, Maryland, Tennessee, Kentucky, Noth\ Carolina\}$

Builder Set Notation: In Builder Set Notation, the following format is used
$\{x : x\ (description)\}$

Here are some examples of sets that written in Builder Set Notation.

$\{x \mid x\ is\ a\ vowel\}$
$\{x \mid x\ is\ a\ great\ lake\}$
$\{x \mid x\ is\ an\ even\ natural\ number\}$

In order to write a set in Builder Set Notation, you must be able to describe the set. A set must be well defined to write in Builder Set Notation.

A set is **well defined** is the elements of the sets are clearly defined.

If a set is well defined, then there should not be any confusion of what the elements are in the set

Examples of well defined sets

$\{1,3,5\}$
$\{m, n, o, p\}$
$\{x \mid x\ is\ a\ whole\ number\}$

Examples of set that are not well defined

$\{x \mid x \text{ is something cool}\}$
$\{x \mid x \text{ is a small dog}\}$

Elements are the members of a given set.

$\in$ *represents is an element of*
$\notin$ *represents is not an element of*

$3 \in \{1,2,3,4,5\}$
$a \in \{a, b, c, d, e\}$

Basic Number Sets

Natural Numbers or Counting Numbers: $N = \{1,2,3,4,5,6,.........\}$
Whole Numbers: $W = \{0,1,2,3,4,5,6,.........\}$
Integers $I = \{.....-3,-2,-1,0,1,2,3,......\}$
Rational Numbers: Q = {x | x is a terminating number or repeating decimal}
Irrational Numbers: J = {x | x is not a terminating number or repeating decimal}
Real Numbers: R = {x | x is a rational number or irrational number}

Practice Problems

Example 1

Write the following set in roster form.

The set of the seven dwarfs

Solution: {*Dopey, Sleepy, Grumpy, Sneezy, Happy, Droopy, Doc* }

Example 2

Write the following set in roster form.

The set of the five great lakes

Solution: {*Huron, Ontario, Michigan, Erie, Superior*}

Example 3

Write the following set in roster form.

The set of all integers

$\{.....-3,-2,-1,0,1,2,3,......\}$

Example 4

Write the following set in Builder Set Notation.

$\{10,15,20,25,30,35\}$

$\{x \mid x\ is\ multiple\ of\ \ five\ between\ 10\ and\ 35\}$

Example 5

Write the following set in Builder Set Notation.

$\{Ohio,\ Utah,\ Iowa\}$

$\{x \mid x\ is\ a\ state\ with\ \ four\ letters\}$

Equivalent Sets

Two sets are equivalent if they have the same number of elements.

Two equivalent sets A and B are denoted by $A \sim B$

Examples of equivalent sets

$\{1,2,3,4\}$
and
$\{a,b,c,d\}$

$\{john, luke, mark, mathew\}$
and
$\{a,b,c,d\}$

Equal Sets

Two sets are equal if their elements ate identical.

Two equal sets A and B are denoted by $A = B$

Example of two equal sets

$\{a, b, c\}$ *and* $\{c, a, b\}$

Or

$\{a, b, c\} \sim \{c, a, b\}$

Example 6

Classify as true or false

1) $2 \in \{1,2,3,4,5\}$

True, 2 is an element of the set $\{1,2,3,4,5\}$

2) $7 \in \{1,2,3,4,5\}$

False, 6 is not in the set $\{1,2,3,4,5\}$

3) $\{1,3,5\} \sim \{a, m, v\}$

True, the two sets have the same number of elements.

4) $\{1\} \in \{1,2,3,4,5\}$

The element $\{1\}$ is not in the set $\{1,2,3,4,5\}$

Section 2.2

Subsets and Improper Subsets

Key Terms

The **empty set** is a set that contains no elements. The empty set is also referred to as the **null set**.

Subsets

A set B is a subset of set C, if every element in B is an element of C. $B \subset C$

Proper Subsets

A set B is a proper subset of C, if every element of B is an element of C and there is at least one element of C that is not in B. $B \subset C$

Example 1

$A = \{1,2,3,4,5\}$
$C = \{1,2,3,4,5,6,7\}$

$Is\ A \subset C?$

Solution: Since every element in the set A is an element of C, A is a subset of C.

Example 2

$Is\ \{4,5,6\}\ a\ subset\ of\ \{0,1,2,3,4,5\}?$

Solution: no, since the element 6 in not in the set $\{0,1,2,3,4,5\}$

Example 2

$Is\ \{4,5,6\}\ a\ proper\ subset\ of\ \{4,5,6\}?$

Solution: The set $\{4,5,6\}$ is a subset of itself, but not a proper subset. Remember that the parent set must have at least one element that is not in the proper subset.

Example 4

List all possible subsets of $\{a, m\}$

Solution: $\phi, \{a\}, \{m\}, \{a, m\}$

Example 5

List all subsets of the set {2,3,4}

Possible subsets

Solution: $\phi, \{2\}, \{3\}, \{4\}, \{2,3\}, \{3,4\}, \{2,4\}\{2,3,4\}$

Example 6

List all subsets of the set {6}

Possible sets: $\phi, \{6\}$

The pattern for subsets

Number of elements	Number of subsets
1	2
2	4
3	8
4	16

Formula to find the number of subsets s of a given set A with n elements

$s = 2^n$

Example 7

How many subsets does a set A with 10 elements have?

$s = 2^n$

$s = 2^{10}$

$s = 1024$

The **universal set** is the set of all possible elements of set used in the problem. Denoted by U

The complement of a set A

The complement of a set A is the set of all elements in the universal that are not elements of the set A.

$A' = \{x \mid x \notin A \ and \ x \in U\}$

Example 8

Find the compliment of each set. The that the universal set is $U = \{0,1,2,3,4,5,6,7,8,9,10\}$

1) $A = \{2,3,4,5\}$

$A' = \{0,1,6,7,8,9,10\}$

2) The odd natural numbers less than 10: $\{1.3.5.7.9\}$

Compliment = $\{0,2,4,6,8\}$

3) $\{1,4,7,8,9,10\}$

Compliment = $\{0,2,3,5,6\}$

Section 2.3

Set Operators

Union and Intersection

Union of Two Sets

The union of two sets is denoted by $A \cup B$ is $A \cup B = \{x \mid x \in A \ or \ x \in B\}$

Intersection of Two Sets

The intersect of two sets is denoted by $A \cap B$ is $A \cap B = \{x \mid x \in A \ and \ x \in B\}$

Example 1

Let $A = \{1,2,3,4,5,6\}$, $B = \{1,3,5,7\}$, $C = \{1,2\}$, $D = \{1,2\}$, and $E = \phi$

1) Is $C \subset A$?
Answer: Yes, every element in C is contained in A

2) Is $\phi \subset A$?
Yes, the empty set is a subset of any nonempty every set.

3) Find $A \cap B$
Answer: $A \cap B = \{1,3,5\}$

4) Find $A \cup B$
Answer: $A \cup B = \{1,2,3,4,5,6,7\}$

5) Find $A \cap C$
Answer: $A \cap C = \{1,2\}$

6) Find $A \cap (B \cap C)$
Answer: $A \cap (B \cap C) = \{1,2,3,4,5,6,7\} \cap (\{1,3,5,7\} \cap \{1,2\}) = \{1,2,3,4,5,6,7\} \cap \{1\} = \{1\}$

7) Find $A \cup (B \cap C)$
Answer:
$A \cup (B \cap C) = \{1,2,3,4,5,6,7\} \cup (\{1,3,5,7\} \cap \{1,2\}) = \{1,2,3,4,5,6,7\} \cup \{1\} = \{1,2,3,4,5,6,7\}$

Example 2

Let $A = \{a,b,c,d\}$, $B = \{a,b,d,e\}$, $C = \{b,c,d\}$, and $D = \{c,d\}$

8) Is $C \subset A$?
Answer: Yes, every element in C is contained in A

9) Is $\phi \subset A$?
Yes, the empty set is a subset of any nonempty every set.

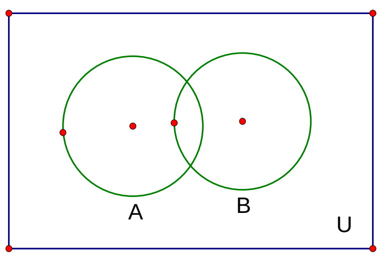
10) Find $A \cap B$
Answer: $A \cap B = \{a,b,d\}$

11) Find $A \cup B$
Answer: $A \cup B = \{a,b,c,d,e\}$

12) Find $A \cap (B \cap C)$
Answer:
$A \cap (B \cap C) = \{a,b,c,d\} \cap (\{a,b,d,e\} \cap \{b,c,d\}) = \{a,b,c,d\} \cap \{b,d\} = \{b,d\}$

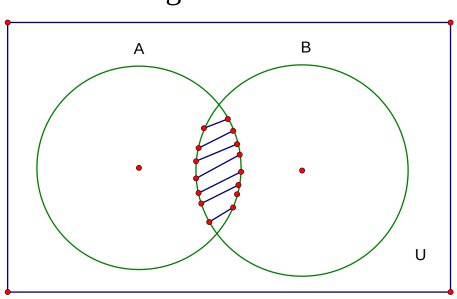
Venn Diagrams

General Venn Diagram for sets A and B



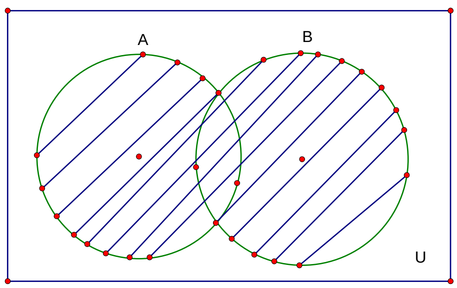
$\mathscr{U}$ = the universal set

The Venn diagram for $A \cap B$



The Venn diagram for $A \cup B$

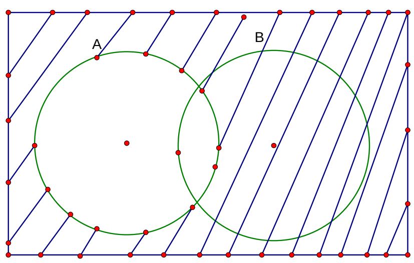
$A \cup B$



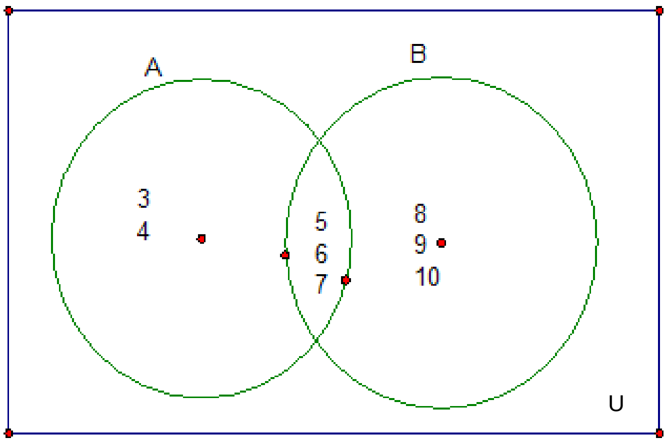
The complement of a set A

The complement of a set A is the set of all elements in the universal that are not elements of the set A.

$A' = \{x \mid x \notin A \ and \ \ x \in U\}$



Example 3



1) Find $A \cap B$
   $A \cap B = \{5,6,7\}$

2) Find $A \cup B$

$A \cup B = \{3,4,5,6,7,8,9,10\}$

3) Find A'

$A' = \{8,9,10\}$

Example 3

Given

$A = \{1,2,3,4,5,6\}, B = \{4,5,6,7,8\}, U = \{1,2,3,4,5,6,7,8,9,10,11,12\}$

Find

1) $A \cup B$

$A \cup B = \{1,2,3,4,5,6,7,8\}$

2) $A \cap B$

$A \cap B = \{4,5,6\}$

3) A'

$A' = \{7,8,9,10,11,12\}$

4) B'

$B' = \{1,2,3,9,10,11,12\}$

5) $(A \cap B)' \cup A'$

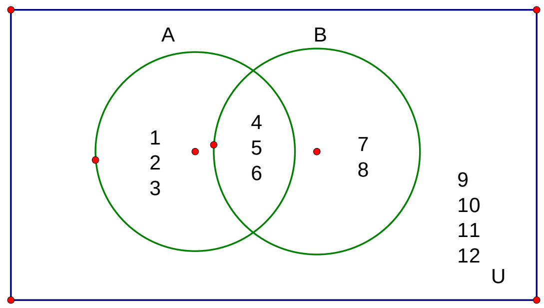
$(A \cap B)' \cup A = (\{1,2,3,4,5,6\} \cap \{4,5,6,7,8\}) \cup \{1,2,3,4,5,6\}'$

$= \{4,5,6\}' \cup \{1,2,3,4,5,6\}'$

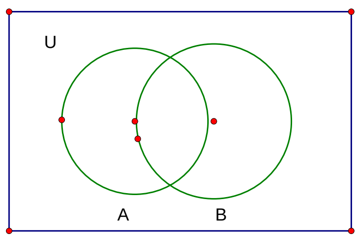
$= \{1,2,3,7,8,9,10,11,12\} \cup \{7,8,9,10,11\}$

$= \{1,2,3,7,8,9,10,11,12\}$

6) Make a Venn diagram of A,B, and U

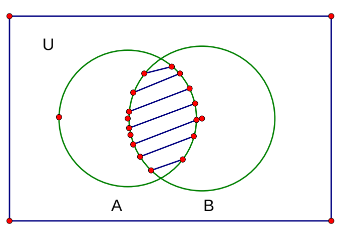


Venn diagrams

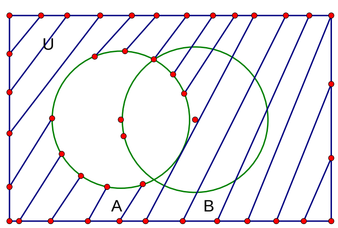


Shade the region corresponding to the indicated set.

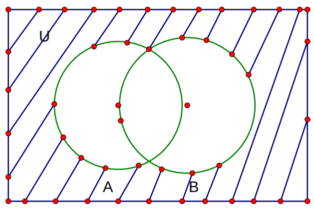
1) $A \cap B$



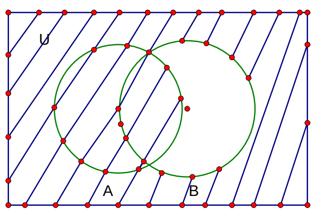
2) A'



3) $A' \cap B'$



4) $A \cup B'$



Section 2.4

Applications of Sets

Definition:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Example 1

Given $n(A) = 340, n(B) = 240, \ and \ n(A \cap B) = 80$, find $n(A \cup B)$

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 340 + 240 - 80 = 580 - 80 = 500$$

Example 2

Given $n(A) = 30, n(B) = 28, \ and \ n(A \cup B) = 50$, find $n(A \cap B)$

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$50 = 30 + 28 - n(A \cap B)$$
$$50 = 58 - n(A \cap B)$$
$$-8 = -n(A \cap B)$$
$$n(A \cap B) = 8$$

Example 3

Given $n(A) = 88, n(B) = 65, \ and \ n(A \cup B) = 120$, find $n(A \cap B)$

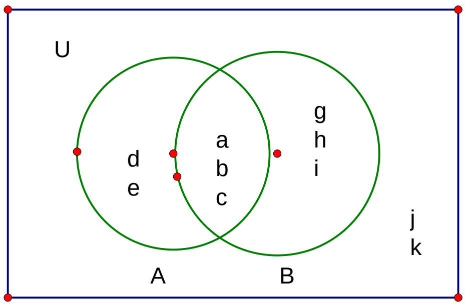
$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$120 = 88 + 65 - n(A \cap B)$$
$$120 = 153 - n(A \cap B)$$
$$-33 = -n(A \cap B)$$
$$n(A \cap B) = 33$$

Cardinality

Definition: Cardinality is the number of elements in a given set

The number of elements in a set A is denoted by $n(A)$

$$A=\{a,b,c,d,e\}, B=\{a,b,c,g,h,i\}, U=\{a,b,c,d,e,f,g,h,,i,j,k\}$$



1) Find $n(A)$

$n(A)=5$

2) Find $n(B)$

$n(B)=6$

3) Find $n(A\cup B)$

$n(A\cup B)=8$

4) Find $n(A\cap B)$

$n(A\cap B)=3$

Rules for the cardinality for the union of two sets

$$n(A\cup B)=n(A)+n(B)-n(A\cap B)$$

Use this formula to find $n(A\cup B)$ in problem 3.

$$n(A\cup B)=n(A)+n(B)-n(A\cap B)=5+6-3=11-3=8$$

Example 5

Let

$U = \{x \mid x \text{ is a state in the United States}\}$
$A = \{x \mid x \in U \text{ and } x \text{ begins with } A\}$
$I = \{x \mid x \in U \text{ and } x \text{ begins with } I\}$
$M = \{x \mid x \in U \text{ and } x \text{ begins with } M\}$
$N = \{x \mid x \in U \text{ and } x \text{ begins with } N\}$
$O = \{x \mid x \in U \text{ and } x \text{ begins with } O\}$

$A = \{Alabama,\ Arkansas,\ Alaska,\ Arizona\}$
$I = \{Iowa,\ Indiana,\ Illinois,\ Idaho\}$
$M = \{Michigan,\ Minnesota,\ Mississippi,\ Missouri,\ Maryland,\ Maine,\ Mon\tan a,\ Massachusetts\}$
$N = \left\{\begin{matrix} Nebraska,\ New\,Jersey,\ New\ Mexico,\ New\,York,\ New\ Hamphere,\ North\ Carolina, \\ North\ Dakota,\ Nevada \end{matrix}\right\}$
$O = \{Ohio,\ Okla\,\text{hom}\,a, Oregon\}$

33) $Find\ \ n(M') = 50 - 8 = 42$
34) $Find\ \ n(A \cup N) = 13$
35) $Find\ \ n(I' \cap O') = 50 - (3 + 4) = 50 - 7 = 43$
36) $Find\ \ n(M \cap I) = 0$

Example 4

Let

$U = \{English,\ Math,\ History, Drama,\ Physics,\ Spanish,\ Philosohpy,\ Chemistry,\ Latin,\ French\}$
$A = \{English,\ History,\ Chemistry,\ Spanish\}$
$B = \{History,\ Math,\ Chemistry,\ French\}$
$C = \{Physics,\ English,\ French,\ Math\}$

1) Find $A \cup B$

$A \cup B = \{English,\ History,\ Chemistry,\ Spanish,\ Math,\ French\}$

2) Find $A \cap B$

$A \bigcap B = \{English,\ Chemistry\}$

3) Find $n(A \cup B)$

$n(A \cup B) = 6$

4) Find $n(A \cap B)$

$n(A \cap B) = 2$

5) Find $n(A) + n(B)$

$n(A) + n(B) = 4 + 4 - 2 = 6$

6) Find $n(A) + n(B) + n(C)$

$n(A) + n(B) + n(C) = 4 + 4 + 4 - 2 - 2 = 8$

Section 2.5

Infinite Sets

Infinite sets and Cardinality

Equivalent Sets

Two sets are equivalent if they have the same number of elements.

Examples of equivalent sets

$$\{1,2,3,4\}$$
$$and$$
$$\{a,b,c,d\}$$

$$\{john, luke, mark, mathew\}$$
$$and$$
$$\{a,b,c,d\}$$

Cardinality

Definition: Cardinality is the number of elements in a given set

One-to-one correspondence

Definition: Two sets are in one-to-one correspondence if each element in the first is paired with exactly one element in the second set, and each element of the second set is paired with exactly one element form the first set

Examples

1) The sets $\{1,2,3,4\}$ *and* $\{a,b,c,d\}$ are in one-to-one correspondence as shown in this diagram.

$$\begin{array}{l} \{1, \;\; 2, \;\; 3, \;\; 4\} \\ \updownarrow \quad \updownarrow \; \updownarrow \; \updownarrow \\ \{a, \; b, \;\; c, \;\; d\} \end{array}$$

2) The sets $\{Viriginia,\ Maryland, North\ Carolina\}$ and $\{Richmond, Annapolis, Raliegh\}$ are in one-to-one correspondence as shown in this diagram.

$$\begin{array}{ccc} \{Viriginia, & Maryland, & North\ Carolina\} \\ \updownarrow & \updownarrow & \updownarrow \\ \{Richmond, & Annapolis, & Raliegh\} \end{array}$$

Cantor's definition of set

A set is **infinite** if we can remove some of its elements without reducing its size.

Countable sets

A set is countable if you establish a one-to-one correspondence form the given set to the natural numbers.

Examples

1) Are the even natural numbers countable?

$$\begin{array}{l} \{2,4,6,8,10,12,.............2n\} \\ \updownarrow\updownarrow\updownarrow\updownarrow\ \updownarrow\ \updownarrow \qquad\qquad \updownarrow \\ \{1, 2,3,4, 5,\ 6,.................n\} \end{array}$$

The even natural can be put in a one-to-one correspondence with the natural numbers by using the mapping $n \leftrightarrow 2n$

2) Are the integers countable?

$$J = \{\ldots\ldots,-4,-3,-2,-1,0,1,2,3,4,\ldots\ldots\ldots\ldots\}$$

The mapping would go as follows:

$0 \leftrightarrow 1$

$1 \leftrightarrow 2$

$-1 \leftrightarrow 3$

$2 \leftrightarrow 4$

$-2 \leftrightarrow 5$

$3 \leftrightarrow 6$

$-3 \leftrightarrow 7$

etc.

Use this mapping

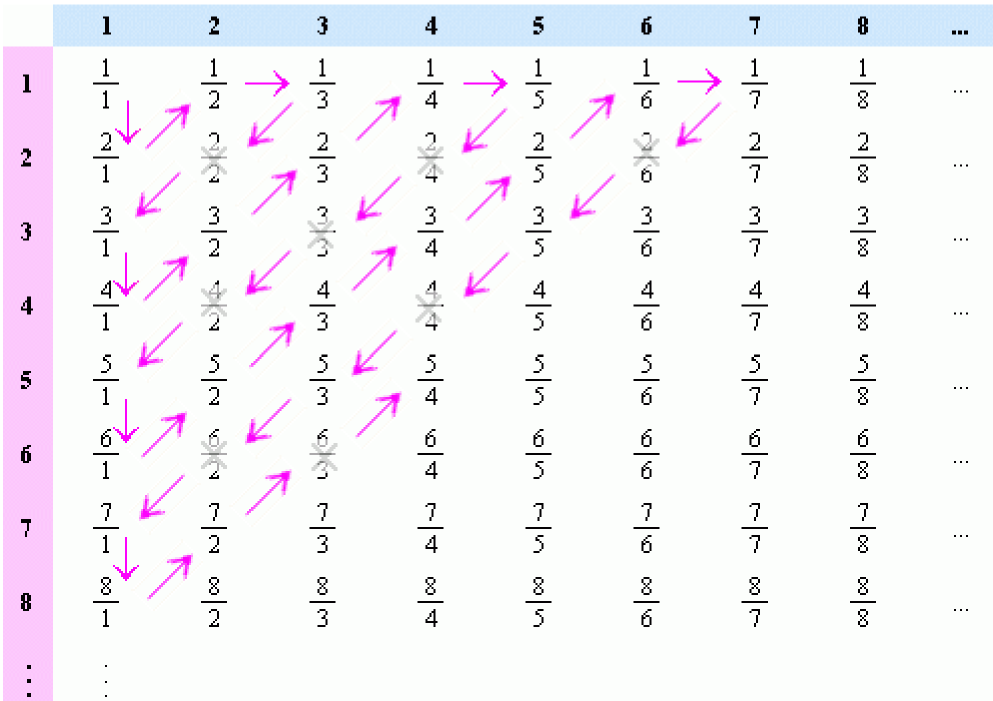
$$n \leftrightarrow \frac{n}{2} \; if \;\; n \; is \; even$$

$$n \leftrightarrow \frac{1-n}{2} \; if \;\; n \; is \; odd$$

Therefore, there exist a one-to-one correspondence between the integers and the natural numbers. Thus, the integers are countable.

3) Are the rational numbers countable?

Look at the following diagram



http://www.homeschoolmath.net/other_topics/rational-numbers-countable.php

This allow the following ordering of numbers

$1 \to 1$

$2 \to 2$

$\frac{1}{2} \to 3$

$\frac{1}{3} \to 4$

$3 \to 5$

......

This shows that each element of the rational number can be paired with one element of the natural numbers. Thus, it is possible to establish a one-to-one correspondence with the natural numbers. This provides an interesting result which is that the rational numbers turn out to countable.