Voting Theory

Majority Rule

If the number of votes n is even, then a majority is $\frac{n}{2}+1$

If the number of votes n is odd, then a majority is $\frac{n+1}{2}$

Example 1

Consider an election with 3 alternatives

Candidate A
Candidate B
Candidate C

There 6 possible rankings ABC, ACB, BAC, BCA, CAB, CBA

Find the winner of the election using majority rule given the results below:

Choices	(ABC)	(ACB)	(BAC)	(BCA)	(CAB)	(CBA)
Number of Votes	5	0	2	1	0	4

To find the winner of the election, consider only the first choice.

Candidate A received 5+0 = 5 first place votes

Candidate B received 2+1 = 3 first place votes

Candidate C received 0+4 = 4 first place votes

Total number of votes n = 12 which is even so use the formula $\frac{n}{2}+1=\frac{12}{2}+1=6+1=7$

Thus, the candidate needs to get 7 votes to win. In this example, all three candidates failed to get seven votes. Therefore, no one won this election.

Plurality Method

Each voter votes for one candidate. The candidate receiving the most votes is declared the winner.

Example 2

Using the voting results form example 1.

Who is the winner using the Plurality method?

Choices	(ABC)	(ACB)	(BAC)	(BCA)	(CAB)	(CBA)
Number of Votes	5	0	2	1	0	4

Solution

Candidate A received 5+0 = 5 votes

Candidate B received 2+1 = 3 votes

Candidate C received 0+4 = 4 votes

Candidate A received the most votes, therefore candidate A is the winner.

Borda Count

Each voter ranks the candidates. If there are n candidates then n points are assigned to the first choice for each voter, with n-1 points for the next choice, and so on. The points are added together for candidate, and the candidate with the most points is declared the winner.

Example 3

Consider the following election for candidates A, B, C, and D

Voter	Ranking
1	B,D,C,A
2	D,C,A,B
3	B,A,C,D
4	B,A,D,C
5	D,A,B,C
6	A,B,C,D

Part 1: Determine the winner by majority rule.

First place votes

A had 1
B had 3
C had none
D had 2

Note: By Majority Rule you would need $\frac{6}{2}+1=3+1=4$ votes to declare a winner.

There is no winner using the majority rule.

Part 2: Determine winner by the Borda Count

Point values for each ranking

1^{st} place (4 points)
2^{nd} place (3 points)
3^{rd} place (2 points)
4^{th} place (1 point)

Use a table to tally the total points for each candidate

Candidate	Voter 1	Voter 2	Voter 3	Voter 4	Voter 5	Voter 6	Total
A	1	2	3	3	3	4	16
B	**4**	**1**	**4**	**4**	**2**	**3**	**18**
C	2	3	2	1	1	2	11
D	3	4	1	2	4	1	15

Since candidate B had the highest total points (18), Candidate B would be the winner.

Pair wise comparison method

The votes rank the candidates by making series of comparisons in which each candidate is compared the other candidates. If a candidate receives more votes than the other candidate then that candidate receives one point. If the candidates receive the same number of votes then they each receive half a point.

Example 4

Pair wise comparison method

Using the same results from the election in examples 1 and 2, determine the winner with pair wise comparison method.

Choices	(ABC)	(ACB)	(BAC)	(BCA)	(CAB)	(CBA)
Number of Votes	5	0	2	1	0	4

Compare A to B

A over B: 5+0+0 = 5

B over A: 2+1+4 = 7

B wins over A
B gets 1 point

Compare B to C

B over C: 5+2+1 = 8

C over B: 0+0+4 = 4

B wins over C
B gets 1 point

Compare A to C

A over C: 5+0+2 = 7

C over A: 1+0+4 = 5

A wins of C
A gets 1 point

Total

A has 1 point
B has 2 points
C has 0 points

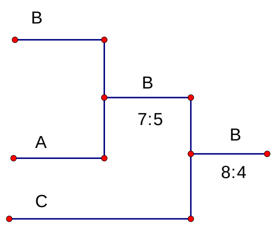
B wins

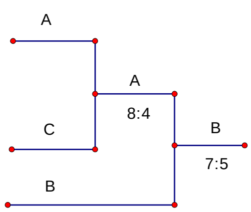
Tournament Method

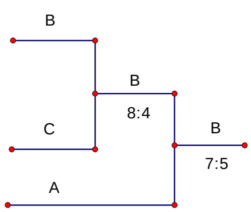
Use the tournament method to find a winner.

Choices	(ABC)	(ACB)	(BAC)	(BCA)	(CAB)	(CBA)
Number of Votes	5	0	3	0	0	4

Compare each candidate by using brackets







B is the winner

Voting Dilemmas

Fair Voting Principles

1) Majority Criterion
2) Condorcet Criterion
3) Monotonicity Criterion
4) Irrelevant Alternatives

Majority Criterion

If a candidate receives a majority of the first place votes, then that candidate should be declared the winner.

Example 1

The South Davis Faculty Association is using the Borda Method to vote for their collective bargaining representative.

Candidate A (All Faculty Association)
Candidate B (American Federation of Teachers)
Candidate C (California Teacher Association)

	ABC	ACB	BAC	BCA	CAB	CBA
Votes	16	0	0	8	0	7

a) Which organization is selected for collective bargaining?

Determined the winner using the Borda Count

	A	B	C
(ABC) 16	3	2	1
(BCA) 8	1	3	2
(CBA) 7	1	2	3

Totals

	A	B	C
ABC(16)	3(16) = 48	2(16) = 32	1(16) = 16
BCA(8)	1(8) = 8	3(8) = 24	2(8) = 16
CBA(7)	1(7) = 7	2(7) = 14	3(7) = 21
Total	63	70	53

The highest Borda Count is from candidate B
Thus, the winner from the Borda Count is candidate B.

Majority Criterion

Since $n = 31$, majority of the votes would be $\frac{n+1}{2} = \frac{31+1}{2} = \frac{32}{2} = 16$

Thus, candidate A had a majority of the votes. Therefore, the Borda Count violates the majority criterion.

Condorcet Criterion

If a candidate is favored when compared one-to-one with every other candidate, then that candidate should be declared the winner.

Example 2

The seniors at Radford High School are voting on where to go on their senior camping trip. They are deciding on Mountain Lake (A), Mount Rodgers (B), Cascade Falls (C), or Bisset Park (D). The results for preferences are:

(DABC)	(ACBD)	(BCAD)	(CBDA)	(CBAD)
120	100	90	80	45

a) Who is the Condorcet candidate?
b) Is there a majority winner? If not is there a plurality winner? Does this violate the Condorcet criterion?
c) Who wins the Borda count? Does this violate the Condorcet criterion?

Solution:

a) Use the Condorcet criterion to find a winner.

Compare A to B

(ACBD) (DABC) $\rightarrow$ A wins

(BCAD) (CBDA) (CBAD) $\rightarrow$ B wins

120+100 = 220 votes for A
90+80+45 = 215 votes for B

Thus, A wins over B

Similarly, we compare the rest of the match ups using one-to-one match ups.

A with C: A: 120+100 = 220 C: 90+80+45 = 215: A wins
A with D: A: 100+90+45 = 235 D: 120+80 = 200: A wins
B with C: B: 120+90 = 210 C: 100+80+45 = 225: C wins
B with D: B: 100+90+80+45 = 315 D: 120: B wins
C with D: C: 100+90+80+45 = 315 D: 120: C wins

Complete a chart for these results:

	A	B	C	D
A	*	A	A	A
B	A	*	C	B
C	A	C	*	C
D	A	B	C	*

Since the column and row headed with A wins in all situations, we can conclude by the Condorcet criterion that A (Mountain Lake) is the winner.

b) Next, use the majority rule.

The first place votes go as follows:

A: 100
B: 90
C: 80+45 = 125
D: 120

In order to get a majority of the vote a candidate would need the following votes.

$n = 435 \ total \ voters$

$$\frac{435+1}{2} = \frac{436}{2} = 218 \, votes \ \ for \ \ a \ \ majority$$

Thus, we can conclude that none of the candidates where able to get a majority of the vote. Therefore, this does violate the Condorcet criterion.

Since the choice C (Cascade Falls) had the most votes, this choice would win the plurality vote. This would be in violation of the Condorcet criterion,

c) Use the Borda count

Table 1 (Point Distribution)

	A	B	C	D
120	3	2	1	4
100	4	2	3	1
90	2	4	3	1
80	1	3	4	2
45	2	3	4	1

Table 2 (Point Totals)

A	B	C	D
3(120) = 360	2(120) = 240	1(120) = 120	4(120) = 480
4(100) = 400	2(100) = 200	3(100) = 300	1(100) = 100
2(90) = 180	4(90) = 360	3(90) = 270	1(90) = 90
1(80) = 80	3(80) = 240	4(80) = 320	2(80) = 160
2(45) = 90	3(45) = 135	4(45) = 180	1(45) = 45
1110	1175	1190	875

Using the Borda count, the winner would be C (Mount Rodgers) This would violate the Condorcet criterion.

Apportionment

The standard divisor

$$d = \frac{total \quad population}{number \ of \ \ seats}$$

Standard Quota

$$q = \frac{state \quad population}{d}$$

Use the data from the 1790 U. S. Census to compute the following values.

a) The standard divisor.

b) The standard quota for each of the U. S states.

Number of seats $N = 105$

State	**1790 population**
Connecticut	237,655
Delaware	59,096
Georgia	82,548
Kentucky	72,677
Maryland	319,728
Massachusetts	475,199
New Hampshire	141,899
New Jersey	184,139
New York	340,241
North Carolina	395,005
Pennsylvania	433,611
Rhode Island	69,112
South Carolina	249,073
Vermont	85,341
Virginia	747,550
Total	3,895,874

a)

$$d = \frac{total \quad population}{number \ of \ \ seats} = \frac{3{,}895{,}874}{105} = 37103.56$$

b) **Solution**

State	**1790 population**	**Standard Quota**
Connecticut	237,655	$q = \frac{237,655}{37,103.56} = 6.41$
Delaware	59,096	$q = \frac{59,096}{37103.56} = 1.59$
Georgia	82,548	$q = \frac{82,548}{37,103.56} = 2.22$
Kentucky	72,677	$q = \frac{73,677}{37,103.56} = 1.99$
Maryland	319,728	$q = \frac{319,728}{37,103.56} = 8.62$
Massachusetts	475,199	$q = \frac{475,199}{37,103.56} = 12.81$
New Hampshire	141,899	$q = \frac{141,899}{37,103.56} = 3.83$
New Jersey	184,139	$q = \frac{184,139}{37,103.56} = 4.97$
New York	340,241	$q = \frac{340,241}{37,103.56} = 9.17$
North Carolina	395,005	$q = \frac{395,005}{37,103.56} = 10.65$
Pennsylvania	433,611	$q = \frac{433,611}{37,103.56} = 11.69$
Rhode Island	69,112	$q = \frac{69,112}{37,103.56} = 1.86$
South Carolina	249,073	$q = \frac{249,073}{37,103.56} = 6.72$
Vermont	85,341	$q = \frac{85,341}{37,103.56} = 2.3$
Virginia	747,550	$q = \frac{747,550}{37,103.56} = 20.16$
Total	3,895,874	