Set and Set Operators

Definition of a set

A **set** is a collection of objects, things or numbers.

The **universal set** is the set of all possible elements of set used in the problem. Denoted by U

Examples

$\{1,2,3,4,5\}$
$\{Ron, John, Mark, Phil\}$
$\{Virginia, West\ Virginia, Maryland, Tennessee, Kentucky, Noth\ Carolina\}$

Elements are the members of a given set.

$\in$ *represents is an element of*
$\notin$ *represents is not an element of*

$3 \in \{1,2,3,4,5\}$
$a \in \{a,b,c,d,e\}$

Roster Notation

$\{a,e,i,o,u\}$
$\{Huron, Ontario, Michigan, Erie, Superior\}$
$\{2,4,6,8,.......\}$

Builder Set Notation

$\{x \mid x\ is\ a\ vowel\}$
$\{x \mid x\ is\ a\ great\ lake\}$
$\{x \mid x\ is\ an\ even\ natural\ number\}$

A set is **well defined** is the elements of the sets are clearly defined.

If a set is well defined, then there should not be any confusion of what the elements are in the set

Examples of well defined sets

$\{1,3,5,7,9,11,13\}$
$\{m, n, o, p, q, r, s\}$
$\{x \mid x \text{ is a whole number}\}$

Examples of set that are not well defined

$\{x \mid x \text{ is something cool}\}$
$\{x \mid x \text{ is a good football team}\}$

Subsets

A set B is a subset of set C, if every element in B is an element of C. $B \subset C$

Proper Subsets

A set B is a proper subset of C, if every element of B is an element of C and there is at least one element of B that is not in C. $B \subset C$

Example

$A = \{1,2,3,4,5\}$
$C = \{1,2,3,4,5,6,7\}$

$Is\ A \subset C?$
Since every element in the set A is an element of C, A is a subset of C.

The Empty Set

The **empty set** is a set that contains no elements. The empty set is also referred to as the **null set**.

Symbol representation ϕ or {}

Example 1

$A = \{1,2,3,4,5\}$
$C = \{1,2,3,4,5,6,7\}$

$Is\ A \subset C?$

Solution: Since every element in the set A is an element of C, A is a subset of C.

Example 2

$Is\ \{4,5,6\}\ a\ subset\ of\ \{0,1,2,3,4,5\}?$

Solution: no, since the element 6 in not in the set $\{0,1,2,3,4,5\}$

Example 3

List all possible subsets of $\{a,c\}$
Solution: $\phi, \{a\}, \{c\}, \{a,c\}$

Example 4

List all subsets of the set {a,b,c}

Possible subsets

Solution: $\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}\{a,b,c\}$

Example 5

List all subsets of the set {4}

Possible sets: $\phi, \{4\}$

The pattern for subsets

Number of elements	Number of subsets
1	2
2	4
3	8
4	16

Formula to find the number of subsets s of a given set A with n elements

$s = 2^n$

Example 6

How many subsets does a set A with 13 elements have?

$s = 2^n$

$s = 2^3$

$s = 8192$

Union and Intersection

Union of Two Sets

The union of two sets is denoted by $A \cup B$ is $A \cup B = \{x \mid x \in A \ or \ x \in B\}$

Intersection of Two Sets

The intersect of two sets is denoted by $A \cap B$ is $A \cap B = \{x \mid x \in A \ and \ x \in B\}$

Example 1

Let A={1,2,3,4,5,6,7},B={1,3,5,7},C={1,2},D={3,4,5,6,7,8}, and E=ϕ

1) Is $C \subset A$?
Answer: Yes, every element in C is contained in A

2) Is $B \subset A$?
Answer: Yes, every element in C is contained in A

3) Is $D \subset A$?
Answer: No, 8 is an element of D and not an element of A?

4) Is $\phi \subset A$?
Yes, the empty set is a subset of any nonempty every set.

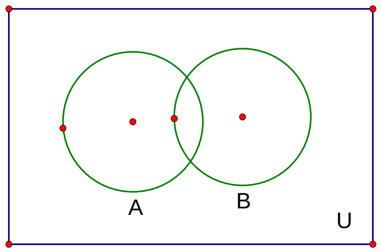
5) Find $A \cap B$
Answer: $A \cap B = \{1,3,5,7\}$

6) Find $A \cup B$
Answer: $A \cup B = \{1,2,3,4,5,6,7\}$

7) Find $A \cap C$
Answer: $A \cap C = \{1,2\}$

Venn Diagrams

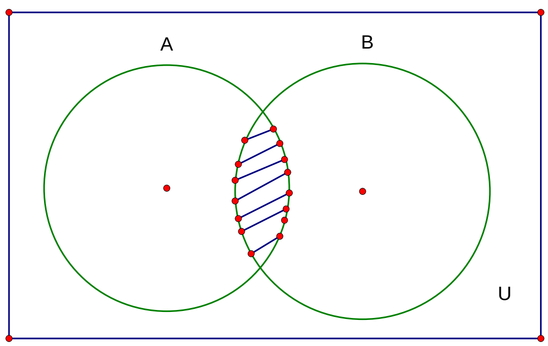
General Venn diagram for sets A and B



$\mathcal{U}$ = the universal set

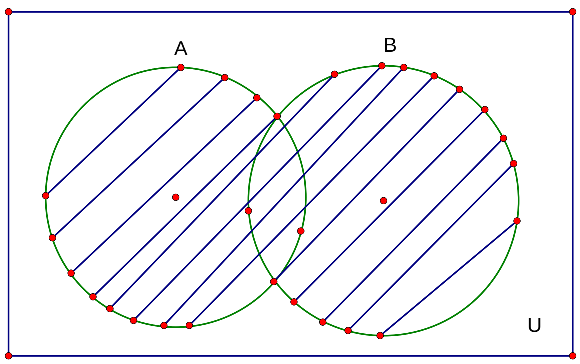
The Venn diagram for $A \cap B$

$A \cap B$



The Venn diagram for $A \cup B$

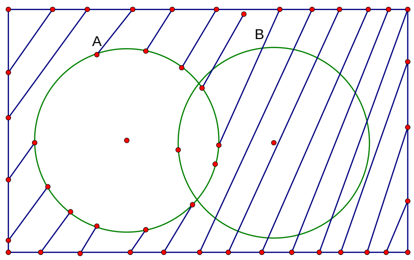
$A \cup B$



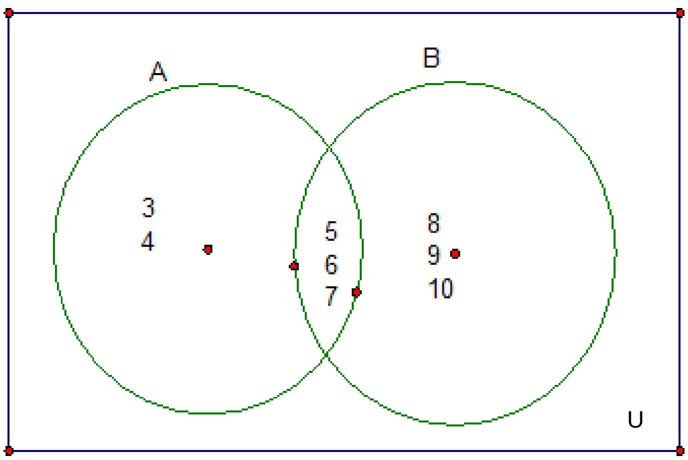
The complement of a set A

The complement of a set A is the set of all elements in the universal that are not elements of the set A.

$A' = \{x \mid x \notin A \ and \ x \in U\}$



Example 2



1) Find $A \cap B$
$A \cap B = \{5,6,7\}$

2) Find $A \cup B$
$A \cup B = \{3,4,5,6,7,8,9,10\}$

3) Find A'
$A' = \{8,9,10\}$

Example 3

Given

$A = \{1,2,3,4,5,6\}, B = \{4,5,6,7,8\}, U = \{1,2,3,4,5,6,7,8,9,10,11,12\}$

Find

1) $A \cup B$

$A \cup B = \{1,2,3,4,5,6,7,8\}$

2) $A \cap B$

$A \cap B = \{4,5,6\}$

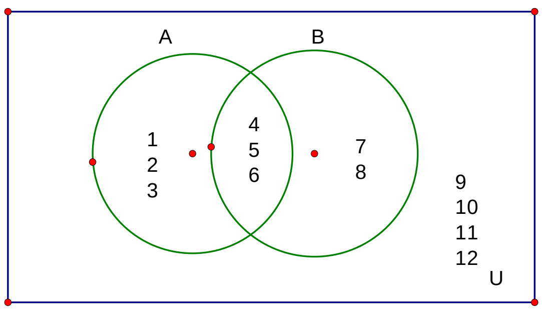
3) A'

$A' = \{7,8,9,10,11,12\}$

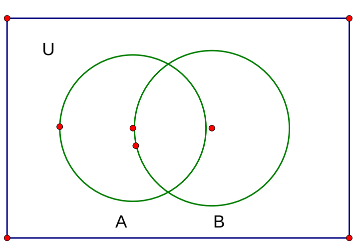
4) B'

$B' = \{1,2,3,9,10,11,10\}$

5) Make a Venn diagram of A,B, and U

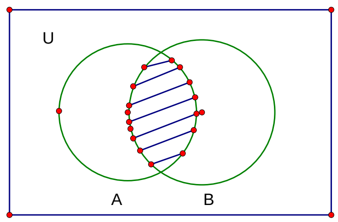


Venn diagrams

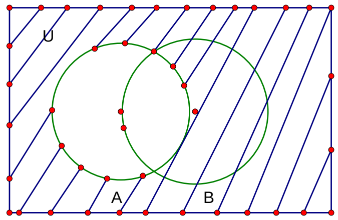


Shade the region corresponding to the indicated set.

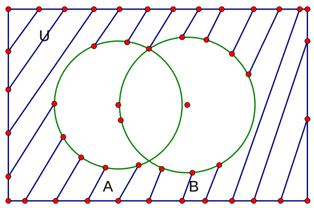
1) $A \cap B$



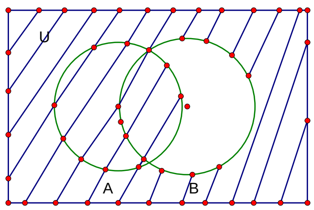
2) A'



3) $A' \cap B'$



4) $A \cup B'$



Problem Set 1

I) Which of the following sets are well defined?

1) $\{1,2,3,4,5,6,7\}$
2) $\{x \mid x\ is\ a\ U\ S\ state\}$
3) $\{x \mid x\ is\ a\ \ fun\ game\}$

II) Write each set in roster form

1) $\{x \mid x\ is\ a\ state\ that\ begins\ with\ the\ letter\ V\}$
2) $\{x \mid x\ is\ a\ vowel\}$

III) Subsets

1) List all subsets of $\{a, i\}$
2) List all subsets of $\{v, p, i\}$
3) List all subsets of $\{4,5,6,7\}$
4) A set of 15 elements would have how many possible subsets

IV) Union and Intersection

$Let\ A = \{a, b, c, d\},\ B = \{b, d, f, h\},\ C = \{h, i, j, k\},\ D = \{e, f, g, h, i\}$

5) Find $A \cup B$
6) Find $A \cap B$
7) Find $B \cup C$
8) Find $A \cap D$
9) Is $C \subset A$?
10) Is $B \subset A$?
11) Make a Venn diagram for sets $A, B,\ and\ U$

V) Union and Intersection

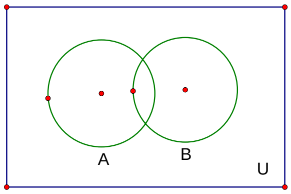
$Let\ A = \{2,4,6,8,10\},\ B = \{3,5,7,9\},\ C = \{2,3,4,5\},\ D = \{1,2,3,4,5,6,7,8,9,10,11,12\}$

$U = \{1,2,3,4,5,6,7,8,9,10,11,12,13,14\}$

12) Find $A \cup B$
13) Find $A \cap B$
14) Find A'
15) Find $(A \cap B)'$
16) Make a Venn diagram for sets $A, B,\ and\ U$

VI) Venn Diagrams

Use the general Venn diagram to answer the following questions



17) Shade the Venn diagram of $A \cup B$
18) Shade the Venn diagram of $A \cap B$
19) Shade the Venn diagram of A'
20) Shade the Venn diagram of $A \cup B'$

Equivalent Sets

Two sets are equivalent if they have the same number of elements.

Examples of equivalent sets

$\{1,2,3,4\}$
and
$\{a,b,c,d\}$

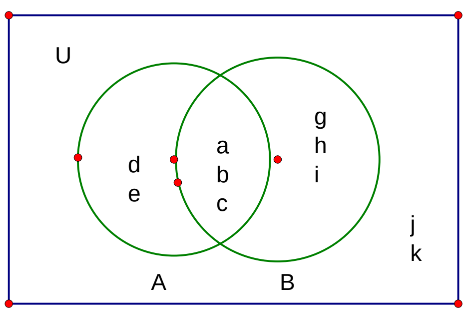
$\{john, luke, mark, mathew\}$
and
$\{a,b,c,d\}$

Cardinality

Definition: Cardinality is the number of elements in a given set

The number of elements in a set A is denoted by $n(A)$

$A=\{a,b,c,d,e\}, B=\{a,b,c,g,h,i\}, U=\{a,b,c,d,e,f,g,h,,i,j,k\}$



1) Find $n(A)$

$n(A)=5$

2) Find $n(B)$

$n(B)=6$

3) Find $n(A\cup B)$

$n(A\cup B)=8$

4) Find $n(A\cap B)$

$n(A\cap B)=3$

Rules for the cardinality for the union of two sets

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Use this formula to find $n(A \cup B)$ in problem 3.

$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 5 + 6 - 3 = 11 - 3 = 8$

Example

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#33-36

Let

$U = \{x \mid x \text{ is a state in the United States}\}$
$A = \{x \mid x \in U \text{ and } x \text{ begins with } A\}$
$I = \{x \mid x \in U \text{ and } x \text{ begins with } I\}$
$M = \{x \mid x \in U \text{ and } x \text{ begins with } M\}$
$N = \{x \mid x \in U \text{ and } x \text{ begins with } N\}$
$O = \{x \mid x \in U \text{ and } x \text{ begins with } O\}$

$A = \{Alabama, Arkansas, Alaska, Arizona\}$
$I = \{Iowa, Indiana, Illinois, Idaho\}$
$M = \{Michigan, Minnesota, Mississippi, Missouri, Maryland, Maine, Mon\tan a, Massachusetts\}$
$N = \begin{Bmatrix} Nebraska, New Jersey, New Mexico, New York, New Hamphere, North Carolina, \\ North Dakota, Nevada \end{Bmatrix}$
$O = \{Ohio, Okla\hom a, Oregon\}$

33) $Find\ n(M') = 50 - 8 = 42$
34) $Find\ n(A \cup N) = 13$
35) $Find\ n(I' \cap O') = 50 - (3 + 4) = 50 - 7 = 43$
36) $Find\ n(M \cap I) = 0$

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Problems 12, 14, 16

$The\ universal\ \ set\ U = \{Monday, Tuesday, Wednesday,\ Thursday,\ Friday,\ Saturday,\ Sunday\}$
$A = \{Monday, Tuesday, Wednesday, Thursday, Friday\}, B = \{Friday, Saturday, Sunday\}$

12) $A \cup B$
$A \cup B = \{Monday, Tuesday, Wednesday,\ Thursday,\ Friday,\ Saturday,\ Sunday\}$

14) A'
$A' = \{Saturday, Sunday\}$

16)
$A \cap B'$
$A \cap B' = \{Monday, Tuesday, Wednesday, Thursday, Friday\} \cap \{Monday, Tuesday, Wednesday, Thursday\}$
$= \{Monday, Tuesday, Wednesday, Thursday\}$

Problem Set 3

From the textbook

Page 435 #7, 8, 9, 11, 13, 15, 27, 37-40

Infinite sets and Cardinality

One-to-one correspondence

Definition: Two sets are in one-to-one correspondence if each element in the first is paired with exactly one element in the second set, and each element of the second set is paired with exactly one element form the first set

Examples

1) The sets $\{1,2,3,4\}$ *and* $\{a,b,c,d\}$ are in one-to-one correspondence as shown in this diagram.

$$\begin{matrix} \{1, & 2, & 3, & 4\} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \{a, & b, & c, & d\} \end{matrix}$$

2) The sets $\{Viriginia,\ Maryland, North\ Carolina\}$ and $\{Richmond, Annapolis, Raliegh\}$ are in one-to-one correspondence as shown in this diagram.

$$\begin{matrix} \{Viriginia, & Maryland, & North\ Carolina\} \\ \updownarrow & \updownarrow & \updownarrow \\ \{Richmond, & Annapolis, & Raliegh\} \end{matrix}$$

Infinite sets

The natural numbers

$N = \{1,2,3,4,5,6,..........\}$

The whole numbers

$W = \{0,1,2,3,4,5,..........\}$

The integers

$J = \{.....,-4,-3,-2,-1,0,1,2,3,4,...........\}$

The rational numbers

$$Q = \left\{ x \mid x = \frac{a}{b}, where\ a\ and\ b \in I \right\}$$

Other infinite sets

The irrational numbers

The real numbers

The real numbers are the rational number and irrational combine as one set.

History of Infinity

Aristotle

Aristotle distinguished between the potential infinite and the actual infinite. Aristotle actually claimed that the natural numbers where potentially infinite because they have no greatest element, but he would not allow them to be actually infinite.

Galileo

Galileo noticed the fact that you could take the natural numbers and remove half of them and remaining set is still as large as the original set.

Galileo's Paradox

If you take the natural numbers and remove the odd elements, the resulting set, the even numbers, is equinumerous to the natural numbers.

In the 1870's **Georg Cantor** discovered that it is possible to determine if two infinite sets are the same size or equinumerous by seeking to find a one-to-one match up between the elements of each set.

Cantor's definition of set

A set is **infinite** if we can remove some of its elements without reducing its size.

Countable sets

A set is countable if you establish a one-to-one correspondence form the given set to the natural numbers.

Examples

1) Are the even natural numbers countable?

$$\{2,4,6,8,10,12,............2n\}$$
$$\updownarrow\updownarrow\updownarrow\updownarrow\ \updownarrow\ \updownarrow\qquad\qquad\quad\updownarrow$$
$$\{1,2,3,4,5,\ 6,................n\}$$

The even natural can be put in a one-to-one correspondence with the natural numbers by using the mapping $n \leftrightarrow 2n$

2) Are the integers countable?

$$J = \{.....,-4,-3,-2,-1,0,1,2,3,4,...........\}$$

The mapping would go as follows:

$$0 \leftrightarrow 1$$
$$1 \leftrightarrow 2$$
$$-1 \leftrightarrow 3$$
$$2 \leftrightarrow 4$$
$$-2 \leftrightarrow 5$$
$$3 \leftrightarrow 6$$
$$-3 \leftrightarrow 7$$
$$etc.$$

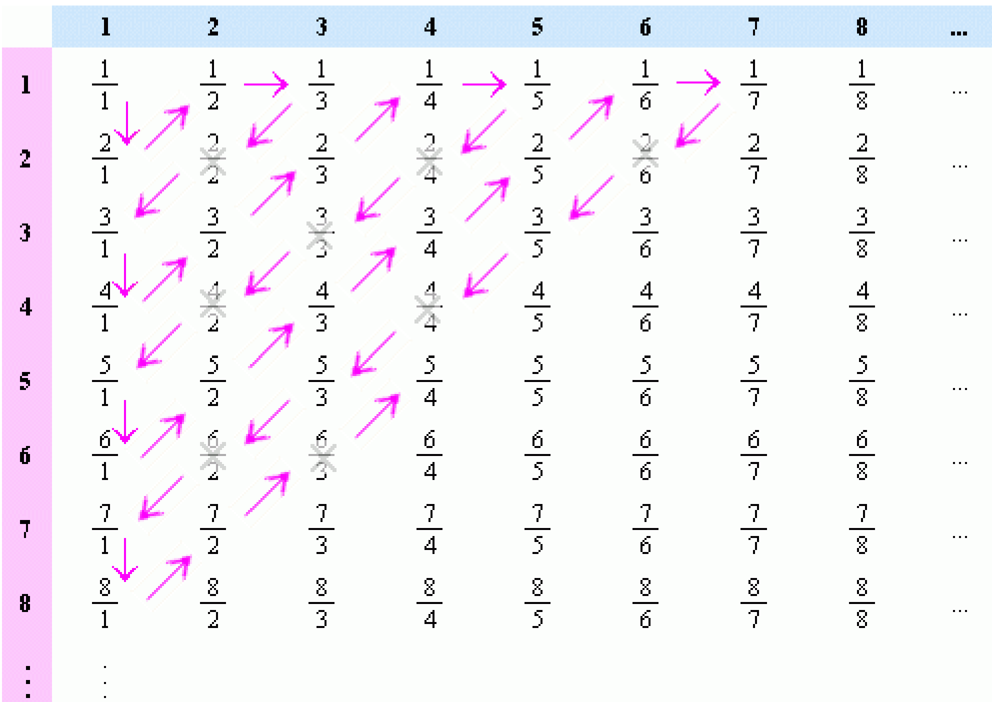
Use this mapping

$$n \leftrightarrow \frac{n}{2}\ if\ \ n\ is\ even$$
$$n \leftrightarrow \frac{1-n}{2}\ if\ \ n\ is\ odd$$

Therefore, there exist a one-to-one correspondence between the integers and the natural numbers. Thus, the integers are countable.

3) Are the rational numbers countable?

Look at the following diagram



http://www.homeschoolmath.net/other_topics/rational-numbers-countable.php

This allow the following ordering of numbers

$1 \to 1$

$2 \to 2$

$\frac{1}{2} \to 3$

$\frac{1}{3} \to 4$

$3 \to 5$

……

This shows that each element of the rational numbers can be paired with one element of the natural numbers. Thus, it is possible to establish a one-to-one correspondence with the natural numbers. This provides an interesting result which is that the rational numbers turn out to countable.

Problem Set 3

1) Give a brief description of Galileo's paradox
2) Briefly discuss Cantor's and Galileo's contributions to set theory

Determine if the following pairs of sets are equivalent.

3) $\{1,2,3,4,5\}$ *and* $\{a, b, c, d, e\}$

4) $\{2,4,6,8\}$ *and* $\{a, b, c, d, e, f\}$

Determine if there is a one-to-one correspondence between the following sets.

5) $\{a, b, c, d, e, f\}$ *and* $\{1,2,3,4,5,6\}$

6) $\{5,6,7,8,9,10,11\}$ *and* $\{1,2,3,4,5,6\}$

7) $\{1,3,5,7,9,11,.........\}$ *and* $\{1,2,3,4,5,6,............\}$

Countable Sets

8) What does it mean for a set to be countable?

9) Explain why the even natural numbers are countable.

10) **(Extra Credit**) Are the rational numbers countable? Explain Why