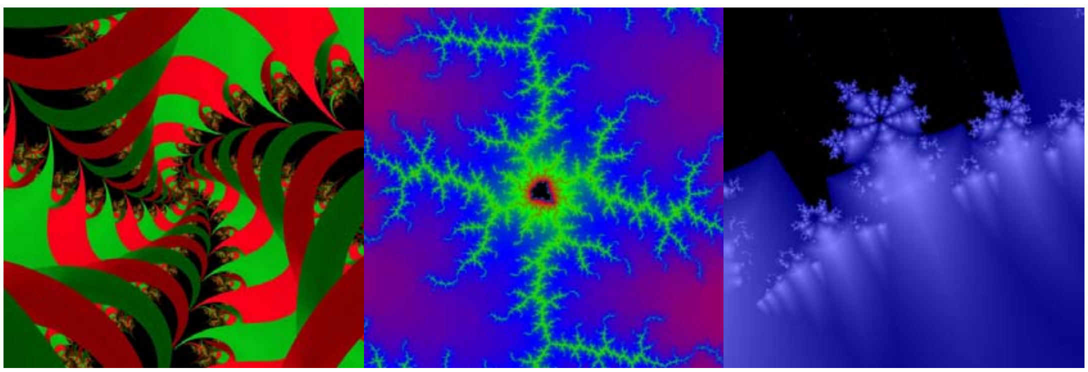
Section 7.6

The theory of ramification is one of pure colligation, for it takes no account of magnitude or position; geometrical lines are used, but these have no more real bearing on the matter than those employed in genealogical tables have in explaining the laws of procreation. J.J. Sylvester

Fractals

To start out this section on fractals we will begin by answering several questions. The first question one might ask is what is a fractal? Usually a **fractal** is defined as a geometric figure that is divided into smaller versions of itself. A second common question that is asked is what does a fractal look like? This question is not as easy to answer because fractals can take on a wide range of patterns and designs, but the common element to all fractals is that they all contain repeating patterns. Below are some computer generated fractals that will give us an idea what a fractal looks like.



All of these picture where generated by Suzanne Alejandre they can view at this website: http://mathforum.org/alejandre/workshops/fractal/fractal3.html

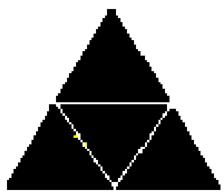
A third question might be is how are fractals created? Usually fractals are made by starting with a general shape which is called an initiator. The initiator is then expanded out into different shapes by using what is called a generator. Here are some examples of how fractals can be generated by using an initiator and a generator.

Example 1

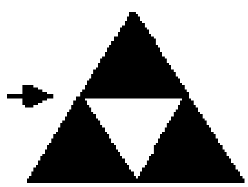
The Sierpinski Triangle

The Sierpinski Triangle is generated by drawing an equilateral triangle and then dividing the triangle in four small equilateral triangles. The next step in the process is shade the outer three triangles and remove the inner triangle. (See the step below)

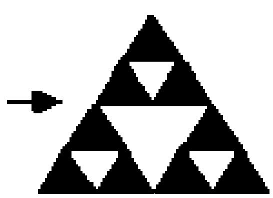
1) Divide the equilateral triangle into four same equilateral triangles as shown:



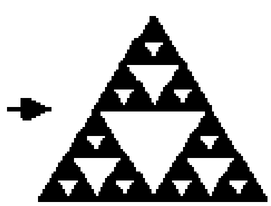
2) Now, Remove the middle triangle.



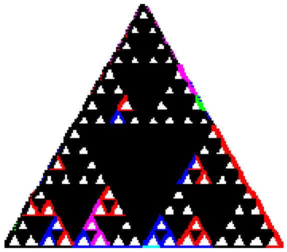
3) Repeat the same pattern by taking the three triangles at the corners and divide those triangles up the as shown for the triangle in step 2



4) Again, repeat the same pattern for the shaded triangles in step 3



By repeating the same processes over and over we can create an interesting fractal called Sierpinski Triangle.



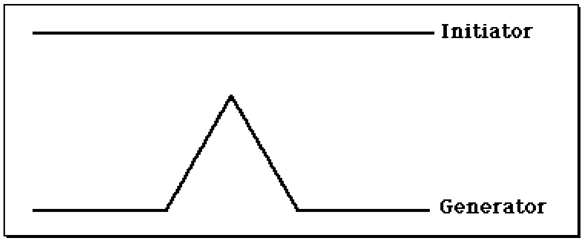
All pictures are courtesy Cynthia Lanius

Example 2

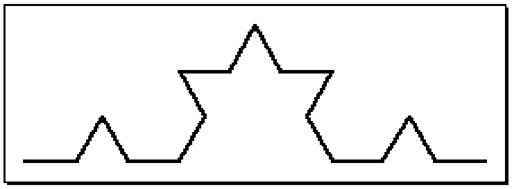
The Koch Curve

When producing the Koch Curve, we start with the given initiator and develop each step using the given generator. In this problem the generator is a line segment and the generator is created by dividing the segment into three equal segments and replacing the middle segment with a hump as shown below in figure 6.2.1. The hump in the middle is formed by segments that equal in length to the outer two segments.

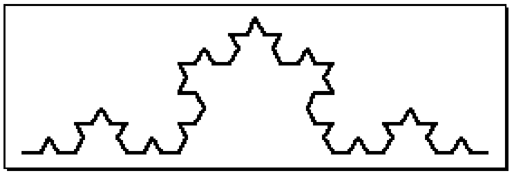
Figure 6.2.1



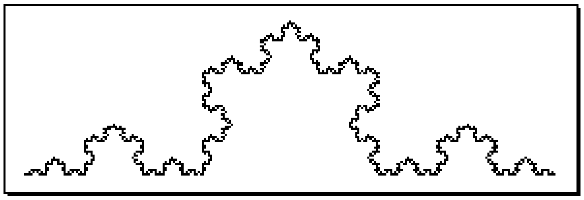
Add the generator segment to each segment will give the following geometric figure



Repeat the same process by adding the same generator segment to each of the newly formed segments.



Repeating this process one more time will give the following fractal



Graphics courtesy Vanderbilt University at:
http://www.vanderbilt.edu/AnS/psychology/cogsci/chaos/workshop/Fractals.html

The dimension of a fractal

The **dimension** of a fractal is used to quantify how densely a strictly self-similar fractal fills the region.

$$d = \frac{\log(N)}{\log(s)}$$

$N = \text{the replacement ratio}$

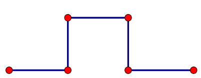
$r = \text{the scaling ratio}$

$$s = \frac{1}{r}$$

Example 1

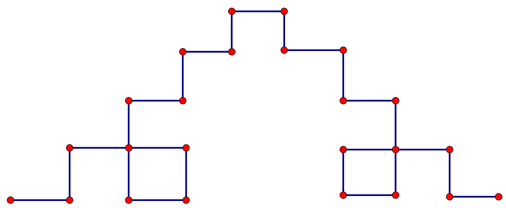
Given the following initiator and generator of a fractal, draw the first iteration of the fractal and find the dimension of the fractal.

Initiator



Generator

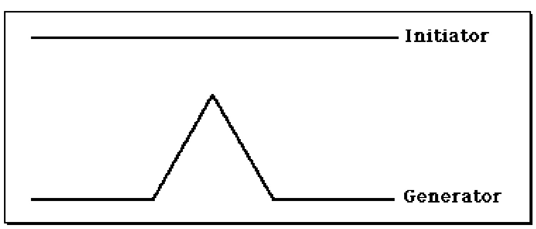
First find the first Iteration, by taking each segment in the generator and repeating that pattern on each segment of the fractal



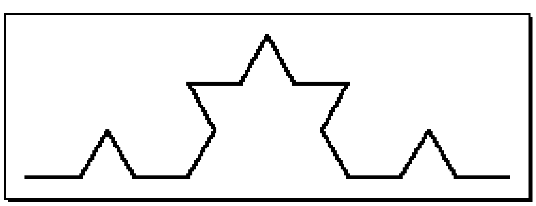
Next, find the dimension of the fractal by finding the values of N, r, and s. The symbol N represents the number of new objects formed in step of the fractal. In this problem, the symbol N is equal to 5. The symbol r is the ratio between the new object formed and the original object. In this problem $r = \frac{1}{3}$ which makes s = 3. Therefore the dimension of the fractal is: $d = \frac{\log(d)}{\log(s)} = \frac{\log(5)}{\log(3)} = \frac{.69897}{.47712} = 1.5$

Example 2

Given the following initiator and generator of a fractal, draw the first iteration of the fractal and find the dimension of the fractal.



First iteration;



Find the dimension of the fractal

$N = 4 \;\; \textit{new objects}$

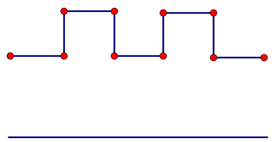
$r = \frac{1}{3}$

$s = 3$

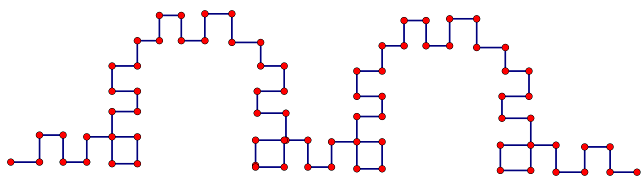
$d = \frac{\log(d)}{\log(s)} = \frac{\log(4)}{\log(3)} = \frac{.6021}{.4771} = 1.26$

Example 3

Given the following initiator and generator of a fractal, draw the first iteration of the fractal and find the dimension of the fractal.



First iteration:



Dimension of the fractal:

$$N = 7 \ \ new\ objects$$

$$r = \frac{1}{5}$$

$$s = 5$$

$$d = \frac{\log(d)}{\log(s)} = \frac{\log(7)}{\log(5)} = \frac{.8451}{.6990} = 1.21$$

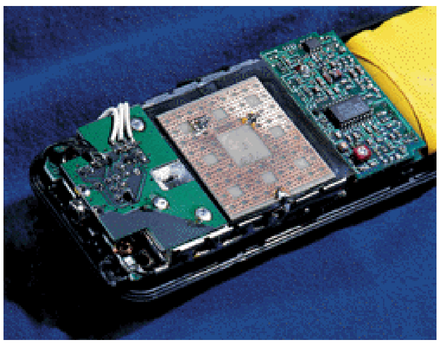
Applications of Fractals

Cellular Phone

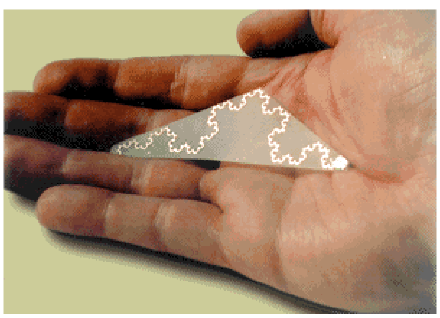
Engineer John Chenoweth discovered that fractal antennas are 25 percent more efficient than rubbery "stubby" antennas. In addition, these types of antenna are cheaper to manufacture and fractal antennas also can operate on multiple bands.

Here are some examples of fractal antennas:

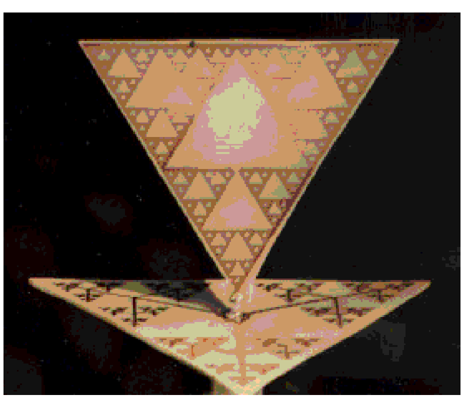
Siepinski's Carpet



Koch Curve



Sierpenski's Triangle



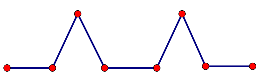
Exercises

1) Given $d = 4 \ and \ N = 6$, find the dimension of the fractal.

2) Given $d = 3 \ and \ N = 5$, find the dimension of the fractal.

3) Given the following initiator and generator, draw the first two iterations and find the dimension of the fractal. Hint $r = \frac{1}{5}$

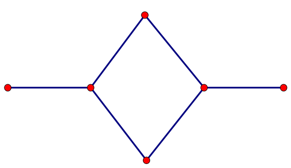
Initiator



Generator

4) Given the following initiator and generator, draw the first two iterations and find the dimension of the fractal. Hint $r = \frac{1}{3}$

Initiator



Generator