Math 151

Section 1.6

Evaluating Limits Analytically

Recall from example 1: $\lim_{x \to 1} x^2 + 1 = 2$

Definition 1: $\lim_{x \to c} f(x) = f(c)$

Thus the limit in the above example can be found by using definition 1,
$\lim_{x \to 1} f(x) = f(1) = 1^2 + 1 = 1 + 1 = 2$

Example 1

Find $\lim_{x \to 3} x^3$

Solution: $\lim_{x \to 3} x^3 = 3^3 = 27$

Example 2

Find $\lim_{x \to -4} x^2 + 5x$

Solution: $\lim_{x \to -4} x^2 + 5x = (-4)^2 + 5(-4) = 16 - 20 = -4$

Example 3

Find $\lim_{x \to 3} \frac{5}{x-3}$

Solution: $\lim_{x \to 3} \frac{5}{x-3} = \frac{5}{3-3} = \frac{5}{0}$ *Undefined*

Limit does not exist

Example 4

Find $\lim_{x\to 1}\sqrt{2x-5}$

Solution: $\lim_{x\to 1}\sqrt{2x-5}=\sqrt{2(1)-5}=\sqrt{2-5}=\sqrt{-3}$ *Undefined*

Limit does not exist

Example 5

Find $\lim_{x\to 3}\frac{x-3}{x^2-9}$

Solution: $\lim_{x\to 3}\frac{x-3}{x^2-9}=\lim_{x\to 3}\frac{x-3}{(x-3)(x+3)}=\lim_{x\to 3}\frac{1}{x+3}=\frac{1}{3+3}=\frac{1}{6}$

In example 5, if you substitute 3 in for x before you reduce the polynomial down, you will get an undetermined form. $\frac{0}{0}$

Example 6

Find $\lim_{x\to 2}\frac{2-x}{x^2-4}$

Solution: $\lim_{x\to 2}\frac{2-x}{x^2-4}=\lim_{x\to 3}\frac{-(x-2)}{(x-2)(x+2)}=\lim_{x\to 3}\frac{-1}{x+2}=\frac{-1}{2+2}=-\frac{1}{4}$

Example 7

Find $\lim_{t \to 1} \dfrac{t^2 + t - 2}{t^2 - 1}$

Solution: $\lim_{t \to 1} \dfrac{t^2 + t - 2}{t^2 - 1} = \lim_{t \to 1} \dfrac{(t+2)(t-1)}{(t-1)(t+1)} = \lim_{t \to 3} \dfrac{t+2}{t+1} = \dfrac{1+2}{1+1} = \dfrac{3}{2}$

Example 8

Find $\lim_{x \to -1} \dfrac{4x - 5}{3 - x}$

Solution: $\lim_{x \to -1} \dfrac{4x - 5}{3 - x} = \dfrac{4(-1) - 5}{3 - (-1)} = \dfrac{-4 - 5}{4} = -\dfrac{9}{4}$

Example 9

Find $\lim_{x \to 2} e^{x-2}$

Solution: $\lim_{x \to 2} e^{x-2} = e^{2-2} = e^{0} = 1$

Example 10

Find $\lim_{x \to \frac{\pi}{2}} \sin x$

Solution: $\lim_{x \to \frac{\pi}{2}} \sin x = \sin\left(\dfrac{\pi}{2}\right) = 1$

Example 11

Find $\lim_{h\to 1} \tan\frac{\pi x}{4}$

Solution:

$$\lim_{h\to 1} \tan\frac{\pi x}{4} = \tan\frac{\pi(1)}{4} = \tan\left(\frac{\pi}{4}\right) = 1$$

Example 12

Find $\lim_{h\to 0} \frac{(h+1)^2 - 1}{h}$

Solution:

$$\lim_{h\to 0} \frac{(h+1)^2 - 1}{h} = \lim_{h\to 0} \frac{(h+1)(h+1) - 1}{h} = \lim_{h\to 0} \frac{h^2 + h + h + 1 - 1}{h} = \lim_{h\to 0} \frac{h^2 + 2h}{h} = \lim_{h\to 0} \frac{h(h+2)}{h}$$
$$= \lim_{h\to 0} h + 2 = 0 + 2 = 2$$

Example 13

Find $\lim_{th\to 0} \frac{\sqrt{t^2+9} - 3}{t^2}$

$$\lim_{th\to 0} \frac{\sqrt{t^2+9} - 3}{t^2} = \lim_{h\to 0} \frac{\left(\sqrt{t^2+9} - 3\right)\left(\sqrt{t^2+9} + 3\right)}{t^2\left(\sqrt{t^2+9} + 3\right)} = \lim_{t\to 0} \frac{\left(\sqrt{t^2+9}\right)^2 + 3\sqrt{t^2+9} - 3\sqrt{t^2+9} - 9}{t^2\left(\sqrt{t^2+9} + 3\right)}$$
$$= \lim_{t\to 0} \frac{t^2 + 9 - 9}{t^2\left(\sqrt{t^2+9} + 3\right)} = \lim_{t\to 0} \frac{t^2}{t^2\left(\sqrt{t^2+9} + 3\right)} = \lim_{t\to 0} \frac{1}{\left(\sqrt{t^2+9} + 3\right)} = \frac{1}{\sqrt{0^2+9} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{3+3} = \frac{1}{6}$$

Example 14

Find $\lim_{x \to 2} \frac{x-2}{x^3-8}$

$$\lim_{x \to 2} \frac{x-2}{x^3-8} = \lim_{x \to 2} \frac{x-2}{(x-2)(x+2x+2^2)} = \lim_{x \to 2} \frac{x-2}{(x-2)(x^2+2x+4)} = \lim_{x \to 2} \frac{1}{x^2+2x+4} = \frac{1}{2^2+2(2)+4} = \frac{1}{12}$$

Limit Laws

1) $\lim_{x \to a} \left(f(x) + g(x) \right) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2) $\lim_{x \to a} \left(f(x) - g(x) \right) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3) $\lim_{x \to a} \left(f(x) \cdot g(x) \right) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

4) $\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

5) $\lim_{x \to a} c \cdot f(x) = c \cdot \lim_{x \to a} f(x)$

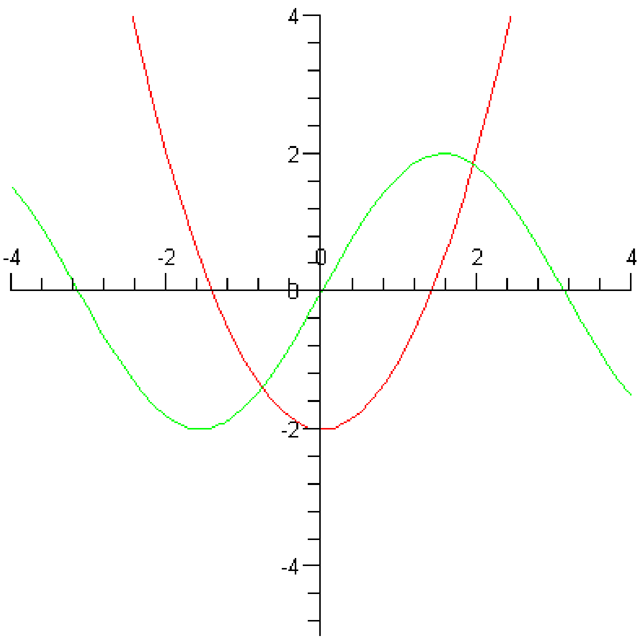
Example 13

Find $\lim_{x \to 0} \frac{\sqrt{5-x}-\sqrt{5}}{x}$

$$\lim_{x \to 0} \frac{\sqrt{5-x}-\sqrt{5}}{x} = \lim_{x \to 0} \frac{\sqrt{5-x}-\sqrt{x}}{x} \cdot \frac{\sqrt{5-x}+\sqrt{5}}{\sqrt{5-x}+\sqrt{5}} = \lim_{x \to 0} \frac{\left(\sqrt{5-x}\right)^2 - \sqrt{5}\sqrt{5-x} + \sqrt{5}\sqrt{5-x} - 5}{x\left(\sqrt{5-x}+\sqrt{5}\right)}$$

$$= \lim_{x \to 0} \frac{5-x-5}{x\left(\sqrt{5-x}+\sqrt{5}\right)} = \lim_{x \to 0} \frac{-x}{x\left(\sqrt{5-x}+\sqrt{5}\right)} = \lim_{x \to 0} \frac{-1}{\sqrt{5-x}+\sqrt{5}} = \frac{-1}{\sqrt{5-0}+\sqrt{5}} = \frac{-1}{2\sqrt{5}}$$

Example 16



$f(x) - green\ line$

$g(x) - red\ line$

Use to the above graph to evaluate each limit

1) Find $\lim_{x \to 0}\left(f(x) + g(x)\right)$

$$\lim_{x \to 0}\left(f(x) + g(x)\right) = \lim_{x \to 0} f(x) + \lim_{x \to 0} g(x) = 0 - 2 = -2$$

2) Find $\lim_{x \to 0}\left(f(x) - g(x)\right)$

$$\lim_{x \to 0}\left(f(x) - g(x)\right) = \lim_{x \to 0} f(x) - \lim_{x \to 0} g(x) = 0 - (-2) = 0 + 2 = 2$$

3) Find $\lim_{x \to 0}\left(f(x) \cdot g(x)\right)$

$$\lim_{x \to 0}\left(f(x) \cdot g(x)\right) = \lim_{x \to 0} f(x) \cdot \lim_{x \to 0} g(x) = 0 \cdot (-2) = 0$$

4) $\lim_{x \to 0}\left(\frac{f(x)}{g(x)}\right)$ $\quad \lim_{x \to 0}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to 0} f(x)}{\lim_{x \to 0} g(x)} = \frac{0}{-2} = 0$