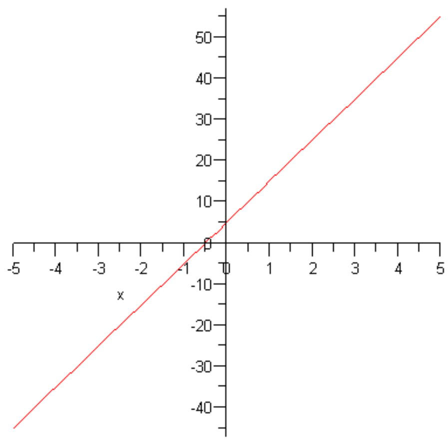
Introduction to Mathematical Modeling

Types of Modeling

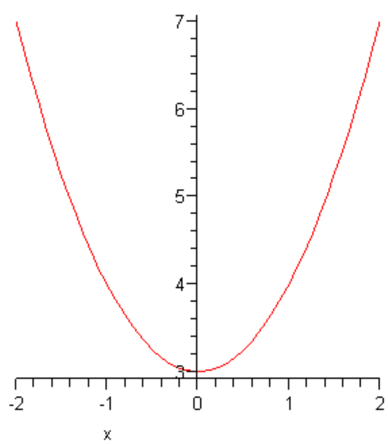
1) Linear Modeling
2) Quadratic Modeling
3) Exponential Modeling
4) Logarithmic Modeling

Each type of modeling in mathematics is determined by the graph of equation for each model. In the next examples, there is a sample graph of each type of modeling

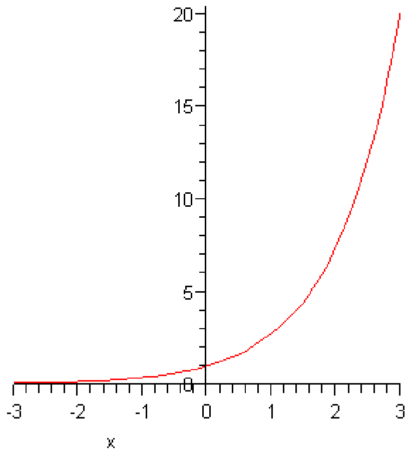
Linear models are described by the following general graph



Quadratic models are described by the following general graph



Exponential models are described by the following general graph



Section 6.2 Linear Modeling

Properties of Linear Functions

The Intercepts of a Function and Equations

The x-intercept is the point where the equation or function intercepts the x axis.

The y-intercept is the point where the equation or function intercepts the y-axis

Find the intercepts of a linear equation

Example 1

Find the x-intercept and y-intercept of each equation.

$$y = \frac{3}{5}x + 2$$

$$\begin{aligned} &Find\ \ the\ \ y - \text{int} \\ &Let\ x = 0 \end{aligned}$$

$$y = \frac{3}{5}(0) + 2 = 0 + 2 = 0$$

$$\begin{aligned} &Find\ \ the\ \ x - \text{int} \\ &Let\ \ y = 0 \end{aligned}$$

$$\begin{aligned} &0 = \frac{3}{5}x + 2 \\ &0 - 2 = \frac{3}{5}x + 2 - 2 \\ &-2 = \frac{3}{5}x \\ &-2 \cdot \frac{5}{3} = \frac{5}{3} \cdot \frac{3}{5}x \\ &\Rightarrow x = -\frac{10}{3} \end{aligned}$$

Example 2

Find the intercept of the given equation

$$4x - 3y = 12$$

$$Find\ \ the\ \ y - \text{int}$$
$$Let\ \ \ x = 0$$

$$\begin{aligned} 4(0) - 3y &= 12 \\ -3y &= 12 \\ \frac{-3y}{-3} &= \frac{12}{-3} \\ y &= -4 \end{aligned}$$

$$Find\ \ the\ \ x - \text{int}$$
$$Let\ \ y = 0$$

$$\begin{aligned} 4x - 3(0) &= 12 \\ 4x &= 12 \\ \frac{4x}{4} &= \frac{12}{4} \\ x &= 3 \end{aligned}$$

The slope of a line

Slope

$$(Slope)\ m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

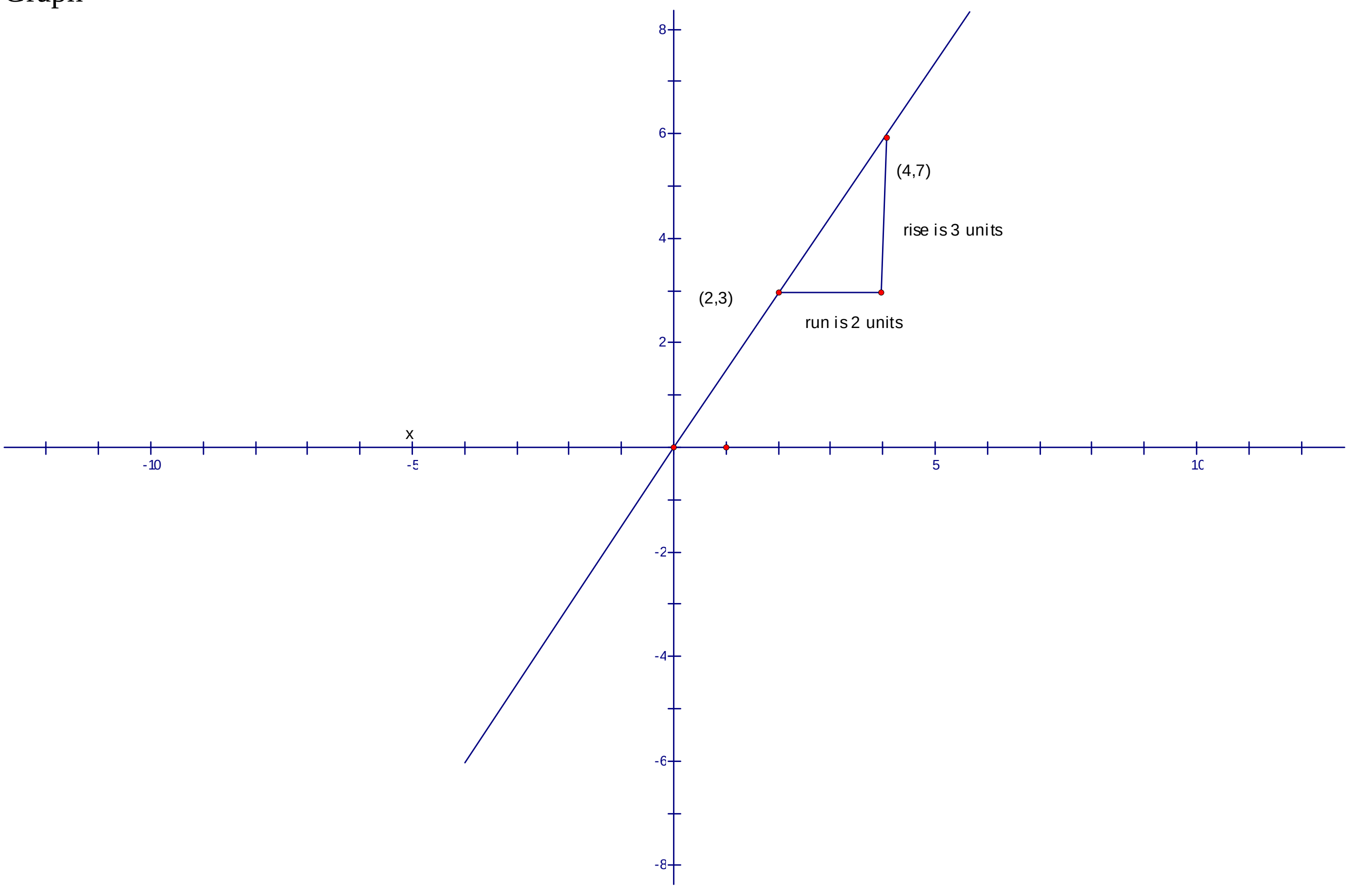
Example 3

Find the slope between the given points

1) (2,3) and (4,6)

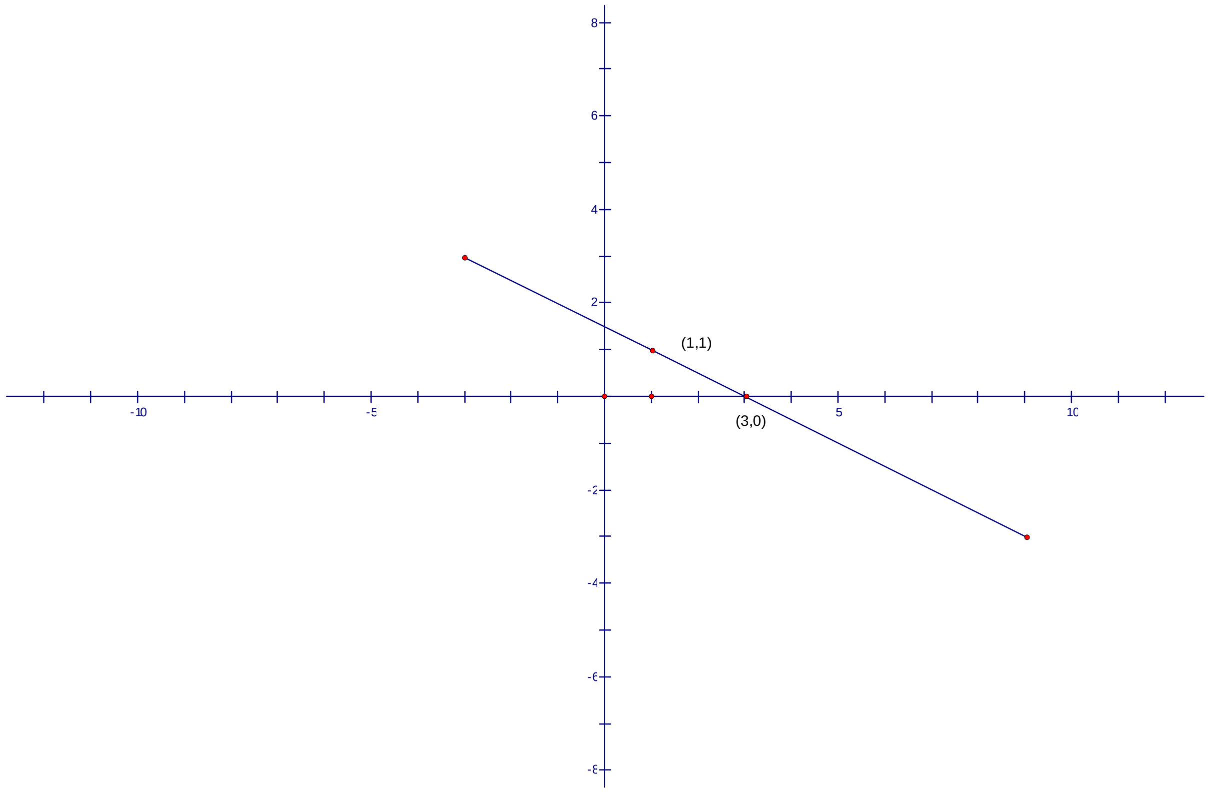
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-3}{4-2} = \frac{3}{2}$$

Graph



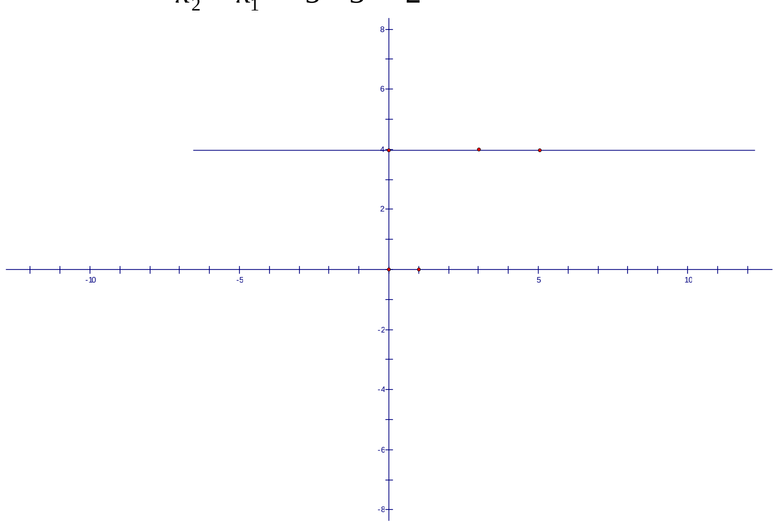
2) (1,1) and (3,0)

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-1}{3-1} = -\frac{1}{2}$$



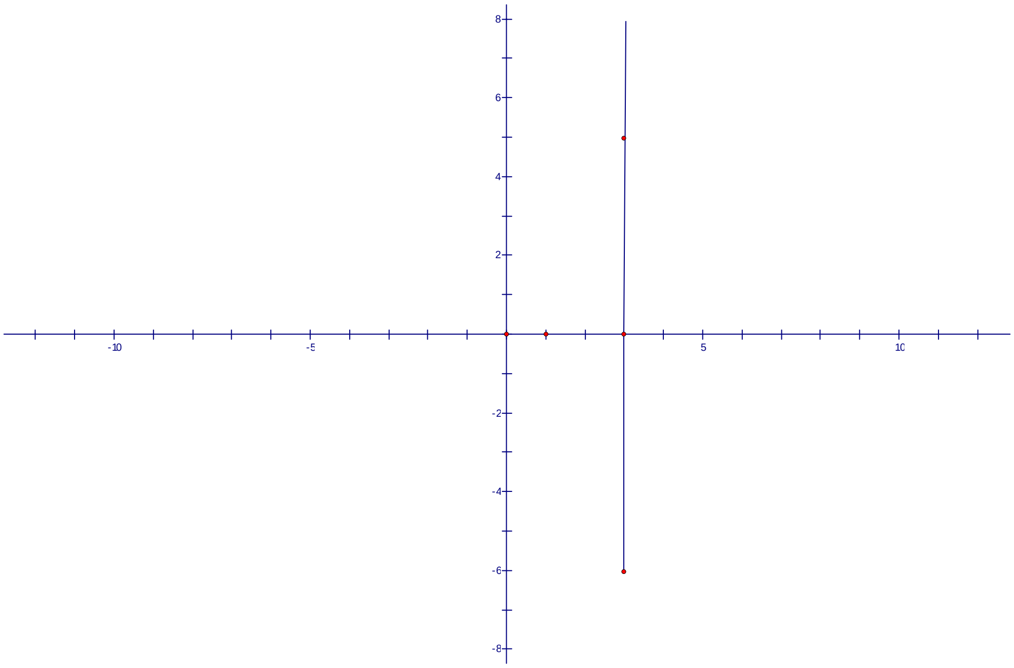
3) (3,4) and (5,4)

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-4}{5-3} = \frac{0}{2} = 0$$



c) (3,5) and (3,8)

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-5}{3-3} = \frac{3}{0}$$



Slope-intercept form of an equation

$y = mx + b$

$m = slope$

$b = y - \text{int}\,ercept$

Example 5

Find the slope and y-intercept, given the equation of the line.

$$2x + y = 40$$
$$2x - 2x + y = -2x + 40$$
$$y = -2x + 40$$
$$m = -2,\ \ b = 40$$

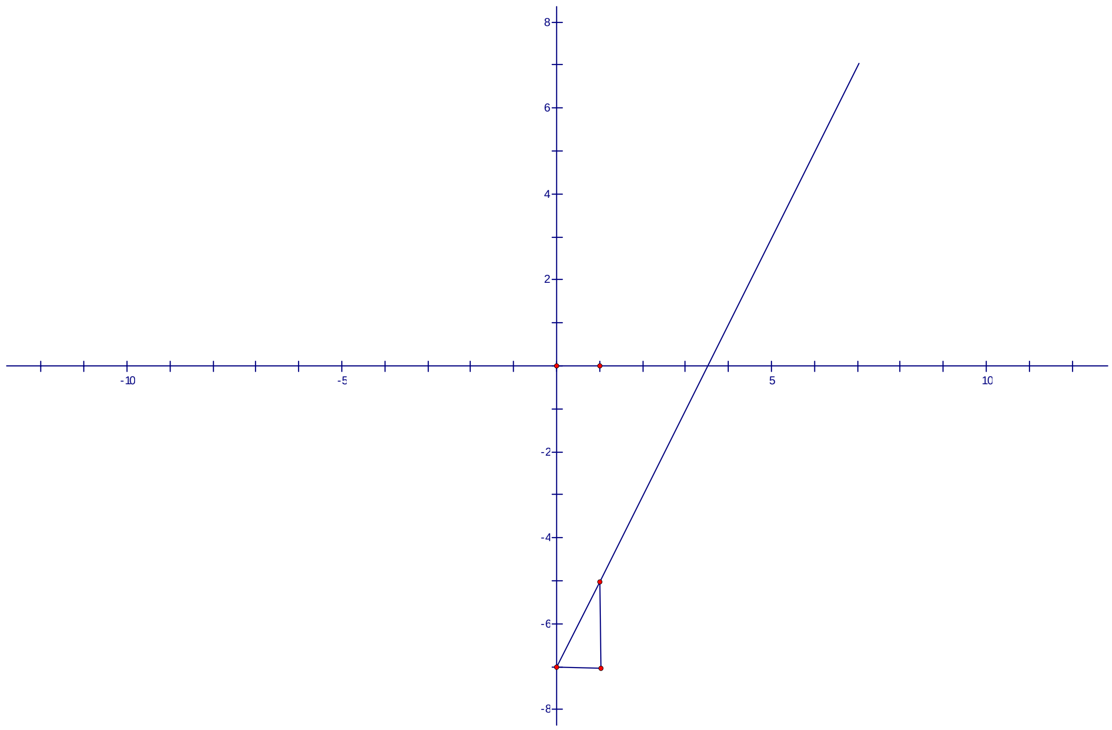
Example 6

Write the equation of the line that passes through the given points. (Use the equation to graph the line. (-3,-4) and (1,4)

Find the slope first: $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-(-4)}{1-(-3)} = \frac{8}{4} = 2$

Next, use the point slope formula and write answer is slope-intercept form with the either point (-3,-4) and (1,4). This example use the point (1,4)

$$y - y_1 = m(x - x_1)$$
$$y - 1 = 2(x - 4)$$
$$y - 1 = 2x - 8$$
$$y - 1 + 1 = 2x - 8 + 1$$
$$y = 2x - 7$$
$$m = 2, b = -7$$



8
6
4
2
-2
-4
-6
-8
-10
-5
5
10

Section 6.3 Linear Modeling

Slope-intercept form of an equation

$y = mx + b$

$m = slope$

$b =$ y-intercept

Example 1

Write the equation of the line that passes through the given points. (Use the equation to graph the line. (-3,-4) and (1,4)

Find the slope first: $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-(-4)}{1-(-3)} = \frac{8}{4} = 2$

Next, use the point slope formula and write answer is slope-intercept form with the either point (-3,-4) and (1,4). This example use the point (1,4)

$y - y_1 = m(x - x_1)$

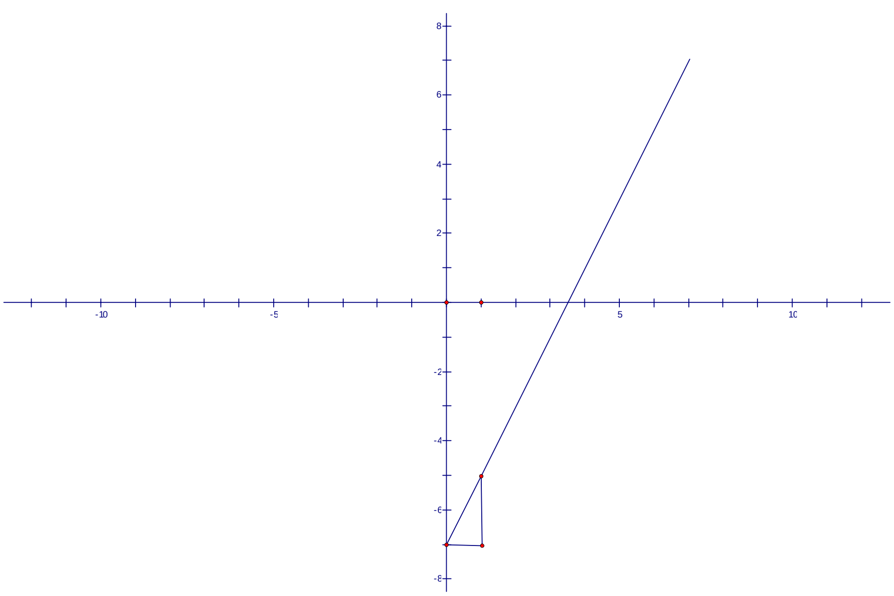
$y - 1 = 2(x - 4)$

$y - 1 = 2x - 8$

$y - 1 + 1 = 2x - 8 + 1$

$y = 2x - 7$

$m = 2, b = -7$



Example 2

Write the equation of the line that passes through the given points. (Use the equation to graph the line. (2,3) and (5,5)

Find the slope first: $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{5-2} = \frac{2}{3}$

Next, use the point slope formula and write answer is slope-intercept form with the either point (2,3) and (5,5). This example use the point (2,3)

$$y - y_1 = m(x - x_1)$$
$$y - 3 = \frac{2}{3}(x-2)$$
$$y - 3 = \frac{2}{3}x - \frac{4}{3}$$
$$y - 3 + 3 = \frac{2}{3}x - \frac{4}{3} + 3$$
$$y = \frac{2}{3}x + \frac{5}{3}$$

Example 3

Find equation of a line given $m = -2$ and the line passes through (1,3)

$$y - y_1 = m(x - x_1)$$
$$y - 3 = -2(x-3)$$
$$y - 3 = -2x + 6$$
$$y - 3 + 3 = -2x + 6 + 3$$
$$y = -2x + 9$$

Example 4

A general building contractor estimates that the cost to build a new home is $40,000 plus $89 per square foot of floor space in the house. Determine a linear function that gives the cost of building a house. Use the model to find the cost to build a 1500 square foot house.

$$Rate = 85$$
$$Initial\ \ Cost = 40{,}000$$

$$f(x) = mx + b$$
$$f(x) = 85x + 40{,}000$$

$$f(1500) = 85(1500) + 40{,}000$$
$$f(1500) = 127{,}500 + 40{,}000 = \$167{,}500$$

Example 5

A plane can travel 1030 miles in two hours. Find a linear model to predict the number of miles that the plane can travel in a certain number of hours. Use the model to predict how far the plane will travel in 5 hours

$$Rate = \frac{1030\ mi}{2\ hr} = 515\frac{mi}{hr}$$

$$f(x) = mx + b$$
$$f(x) = 515x + 0 = 515x$$

$$f(5) = 515(5) = 2575\ miles$$

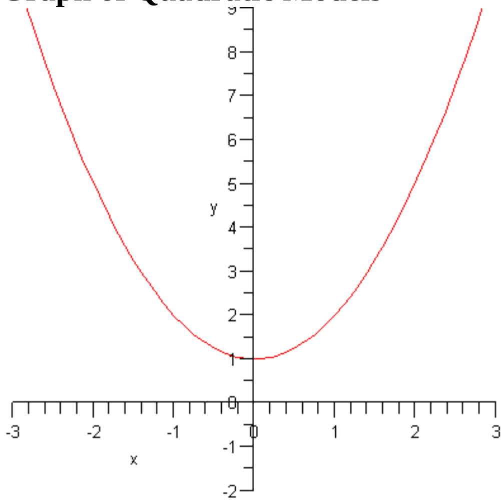
Example 6

A shoe store discovered that if sold a pair of running shoes at $60 per pair they sold 20 pairs of shoes in one week. The store also discovered that if they sold the shoes for $45 per pair, they ended up selling 45 pairs of shoes in one week. Find a model that will find the number of shoes sold a week at a price of x dollars per pair.

$$m = \frac{\$60 - \$45}{45 - 20} = \frac{\$15}{25} = \$.60$$

Quadratic Models

Graph of Quadratic Models



The parabola

A **quadratic function** is a function where the graph is a parabola and an equation of the

Form: $y = ax^2 + bx + c$ where $a \neq 0$

The x coordinate vertex is given by the equation: $x = -\dfrac{b}{2a}$

Example 1

Find the vertex of the graph of the equation.

$$y = x^2 - 6x + 5$$
$$a = 1$$
$$b = -6$$
$$c = 5$$
$$x = -\frac{b}{2a} = -\frac{-6}{2(1)} = 3$$
$$y = 3^2 - 6(3) + 5 = 9 - 18 + 5 = -4$$
$$(3, -4)$$

Example 2

Find the intercepts of the following equation.

$$y = x^2 - 6x + 5$$

Find the x-intercept by letting y = 0

$$x^2 - 6x + 5 = 0$$
$$(x-5)(x-1) = 0$$
$$x - 5 = 0 \quad or \quad x - 1 - 0$$
$$x = 5 \qquad\qquad x = 1$$

Find the y-intercepts by letting x = 0

$$y = 0^2 - 6(0) + 5 = 5$$

Example 3

Find the intercepts of the following equation.

$$y = -x^2 - 3x - 2$$

Find the x-intercept by letting y=0

$$0 = -x^2 - 3x - 2$$
$$x^2 + 3x + 2 = 0$$
$$(x-2)(x-1) = 0$$
$$x - 2 = 0 \quad or \quad x - 1 = 0$$
$$x = 2 \qquad\qquad x = 1$$

Find the y-intercept by letting x = 0

$$y = -0^2 - 0 - 2 = 0 - 0 - 2 = -2$$

Example 4

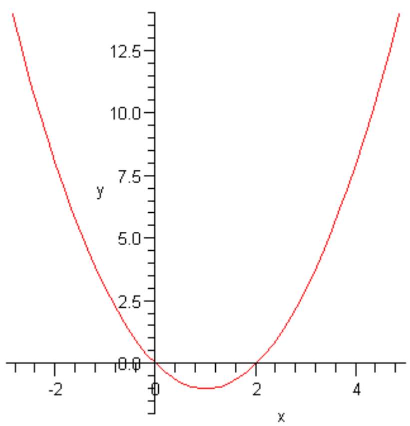
Find the vertex and x-intercepts, and then make a sketch of the parabola.

$$y = x^2 - 2x$$
$$a = 1, b = -2$$
$$x = -\frac{-2}{2(1)} = \frac{2}{2} = 1$$

x-intercepts

$$x^2 - 2x = 0$$
$$x(x-2) = 0$$
$$x = 0 \ \ or \ \ x - 2 = 0$$
$$x = 0 \ or \ \ x = 2$$
$$(0,0) \ and \ (2,0)$$

Graph



Example 5

Find the vertex and x-intercepts, and then make a sketch of the parabola.

$y = x^2 - 3x$

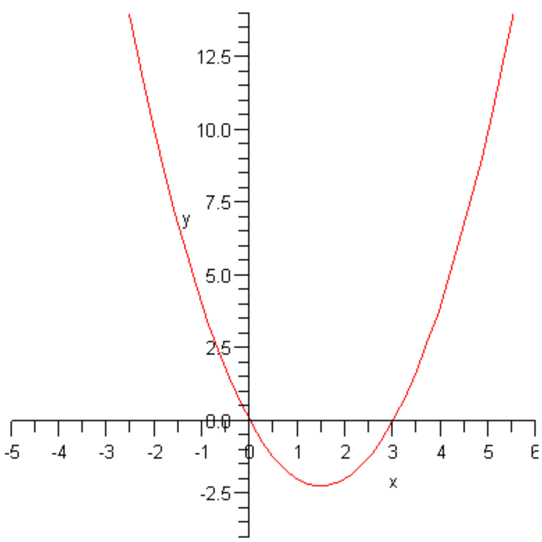
Vertex

$x = -\frac{-3}{2(1)} = \frac{3}{2}$

x-intercepts

$x^2 - 3x = 0$
$x(x-3) = 0$
$x = 0 \ or \ x - 3 = 0$
$x = 0 \quad x - 3 = 0$
$x = 3$
$(0,0) \ and \ (3,0)$

Graph of the function



Example 6

vertex :

$$x = -\frac{-(-4)}{2(1)} = 2$$

$$y - coordinate : y = 2^2 - 4(2) + 3 = -1$$

x-intercepts

$$x^2 - 4x + 3 = 0$$

$$(x-3)(x-1) = 0$$

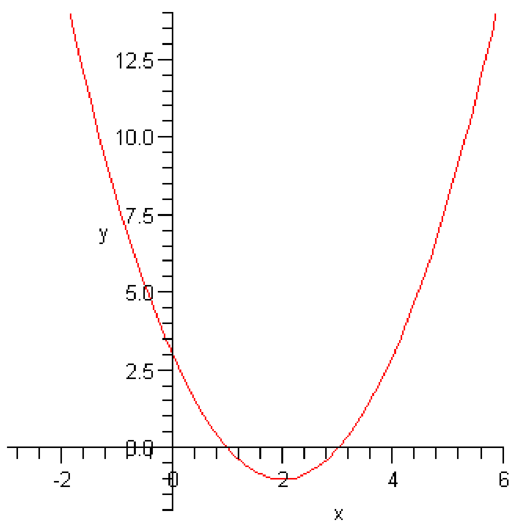
$$x - 3 = 0 \quad or \quad x - 1 = 0$$

$$x - 3 + 3 = 0 + 3 \quad x - 1 + 1 = 0 + 1$$

$$x = 3 \qquad x = 1$$

$(1,0)$ *and* $(3,0)$

Graph



Example 7

Find the vertex and x-intercepts, and then make a sketch of the parabola.

$$y = x^2 - 3$$

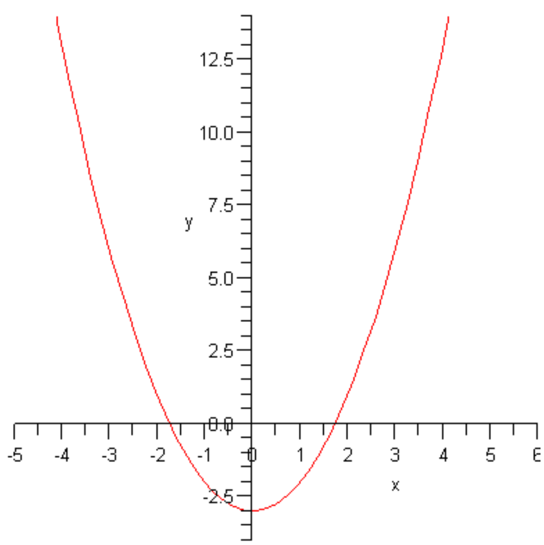
$$a = 1, c = -3$$

$$x = -\frac{0}{2(1)} = -\frac{0}{2} = 0$$

x-intercepts

$$x^2 - 3 = 0$$
$$x^2 = 3$$
$$\sqrt{x^2} = \sqrt{3}$$
$$x = \pm\sqrt{3}$$
$$(\sqrt{3},0) \textit{ and } (-\sqrt{3},0)$$

Graph



Example 8

The path of a ball thrown by a boy is given in yards by the equation $y = -.04x^2 + 1.5x$ where x is the horizontal distance the ball travels and y is the height of the ball. Find the maximum height of the ball in yards.

Find the vertex of the ball

$$x = -\frac{1.5}{2(-.04)} = \frac{1.5}{.08} = 18.75$$

$$y = -.04(18.75)^2 + 1.5(18.75) = -14.1 + 28.1 = 14 \ \textit{yards}$$

Example 9

The path of a cannon ball is given in feet by the equation $y = -.1x^2 + 6.0x$ where x is the horizontal distance the ball travels and y is the height of the cannon ball. Find the maximum height of the cannon ball in feet.

Find the vertex of the cannon ball.

$$x = -\frac{6.0}{2(-.1)} = -\frac{6.0}{-.2} = 30$$

$$y = -.1(30)^2 + 6(30) = -90 + 180 = 90 \ \textit{feet}$$

Example 10

A pool is treated with a chemical to reduce the number algae in the pool t days after the treatment. The algae concentration in the pool can be approximate by the function

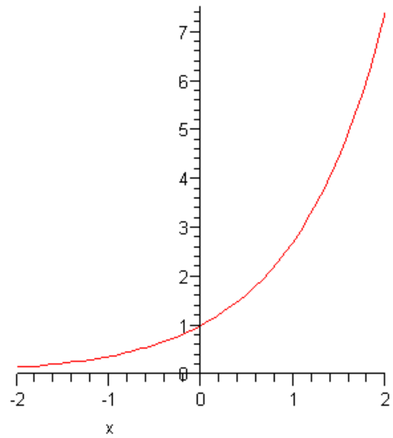
$$A(t) = 30t^2 - 300t + 450$$

How many days after treatment will produce a least amount of algae?

$$x = -\frac{b}{2a} = -\frac{-300}{2(30)} = \frac{300}{60} = 5 \ \textit{days}$$

Section 6.5

Exponential Models



An example of exponential model:

$f(x) = 4^x$

Find $f(2)$

$f(2) = 4^2 = 16$

Find $f(-3)$

$f(-3) = 4^{-3} = \frac{1}{4^3} = \frac{1}{64}$

Find $f(-1)$

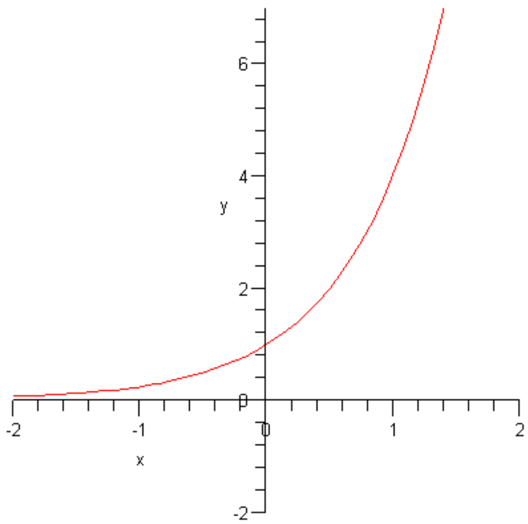
$f(-1) = 4^{-1} = \frac{1}{4^1} = \frac{1}{4}$

Now let's graph the function: $f(x) = 4^x$

Example 1

Graph $y = 4^x$

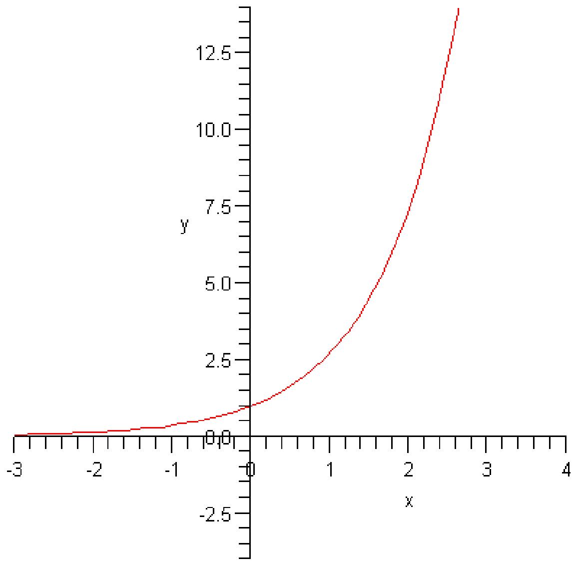
x	$y = 4^x$
-2	$y = 4^{-2} = .08$
-1	$y = 4^{-1} = .25$
0	$y = 4^0 = 1$
1	$y = 4^1 = 4$
2	$y = 4^2 = 16$



Example 2

Graph $y = e^x$

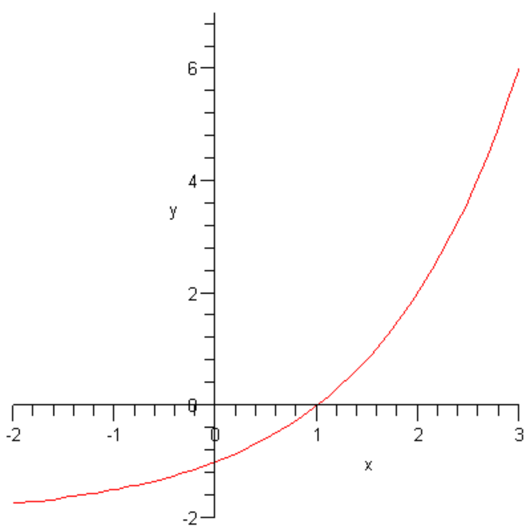
x	y
-2	$y = e^{-2} = .14$
-1	$y = e^{-1} = .37$
0	$y = e^0 = 1$
1	$y = e^1 = 2.7$
2	$y = e^2 = 7.3$



Example 3

Graph $y = 2^x - 2$

x	$y = 2^x - 2$
-2	$y = 2^{-2} - 2 = -1.75$
-1	$y = 2^{-1} - 2 = -1.5$
0	$y = 2^0 - 2 = -1$
1	$y = 2^1 - 2 = 0$
2	$y = 2^2 - 2 = 2$



Compound Interest Formula

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$P = \textit{principle}$

$r = \textit{rate}$

$t = \textit{time}$

$n = \textit{number of compoundings in a year}$

Example 4

Suppose somebody invests $10,000 at 3% per year in an account that compounds interest monthly for 5 years. Find the balance in the account after 5 years.

$$A = P\left(1+\frac{r}{n}\right)^{nt} = 10{,}000\left(1+\frac{.03}{12}\right)^{12(5)} = 10{,}000(1+.0025)^{60} = 10000(1.0025)^{60} = \$11616.17$$

Example 5

The population of Virginia is modeled by the function $P = 7{,}600{,}000(1.011)^{.5t}$ where t is the time in years. Use the model to predict the population of Virginia in 10 years.

$$P = 7{,}600{,}000(1.011)^{.5(10)}$$

$$P = 7{,}600{,}000(1.011)^{5}$$

$$P = 8027298$$

Example 6

The certain population of 2500 bacteria is modeled by the function $P = 2500(e)^{.09t}$ where t is time in months. Use the population to predict the population of bacteria in 24 months.

$P = 2500e^{.09t}$

$P = 2500e^{.09(24)}$

$P = 2500e^{2.16}$

$P = 21678$

Example 7

The population of Chicago is modeled by the function $P = 2{,}700{,}000(1.01)^{25t}$ where t is the time in years. Use the model to predict the population of Chicago in 10 years.

$P = 2{,}700{,}000(1.01)^{\backslash.25(10)}$

$P = 2{,}700{,}000(1.01)^{25}$

$P = 2{,}706{,}724$

Section 6.6
Logarithms and Logarithmic Models

Definition of a logarithm

$\log_b a = x \Leftrightarrow b^x = a$

Example 1

Write each logarithm expression as an exponent expression.

a) $\log_5 125 = 3$

Solution: $5^3 = 125$

b) $\log_6 36 = 2$

Solution: $6^2 = 36$

c) $\log_6\left(\frac{1}{216}\right) = -3$

Solution: $6^{-3} = \frac{1}{216}$

Example 2

Write each exponential expression as an logarithmic expression.

a) $2^5 = 32$

Solution: $\log_2 32 = 5$

b) $4^4 = 256$

Solution: $\log_4 256 = 4$

Example 3

Evaluate each logarithm

a) Find $\log_4 64$

$\log_4 64 = x$

$4^x = 64$

$4^x = 4^3$

$x = 3$

b) Find $\log_3\left(\frac{1}{81}\right)$

$\log_3\left(\frac{1}{81}\right) = x$

$3^x = \frac{1}{81}$

$3^x = \frac{1}{3^4}$

$3^x = 3^{-4}$

c) Find $\log_{10} 10000$ *or* $\log 10000$

$\log_{10} 10000 = x$

$10^x = 10000$

$10^x = 10^4$

$x = 4$

Example 4

Solve the equation for x

$$\log_3 x = 5$$
$$3^5 = x$$
$$x = 243$$

The natural logarithm

$$\ln a = x \Leftrightarrow e^x = a$$
$$Note: \ \ln a = \log_e a$$

Example 5

Write the logarithm expression as an exponent expression

$$\ln 7.39 = 2$$

Solution: $e^2 = 7.39$

pH Models and Logarithm scales

Chemistry uses logarithm to determine the pH of liquid. The pH of a liquid measures the acidity or alkalinity of a liquid. A liquid with a pH of 1 is a very strong acid and a liquid with a pH of 14 is a very strong base. Specifically, the acidity of a substance is a function of its hydrogen-ion concentration. The pH of substance can be determined by the taking the log of its hydrogen concentration. H^+

$$pH = -\log\left[H^+\right]$$

Example 6

Find the pH of a sample of orange juice that has a hydrogen-ion concentration of $3.0 \ x \ 10^{-4} \ mole \ per \ liter$

$$pH = -\log\left[H^+\right] = -\log\left[3.0 \ x \ 10^{-4}\right] =$$

World Oil Supply

Example 7

The time it will take the world's oil supply to be depleted can be modeled by the following formula where r is the estimated oil reserves in billions of barrels.

$$T(r) = 14.29\ln(0.0041r + 1)$$

a) Use the model to find out how much time it will take to use 500 billions barrels.

$$T(r) = 14.29\ln(0.0041r + 1) = 14.29\ln(0.0041(500) + 1) = 14.29(\ln(2.05) + 1) = 14.29(1.718) \approx 25 \ years$$

b) How many barrels of oil are necessary to last 40 years?

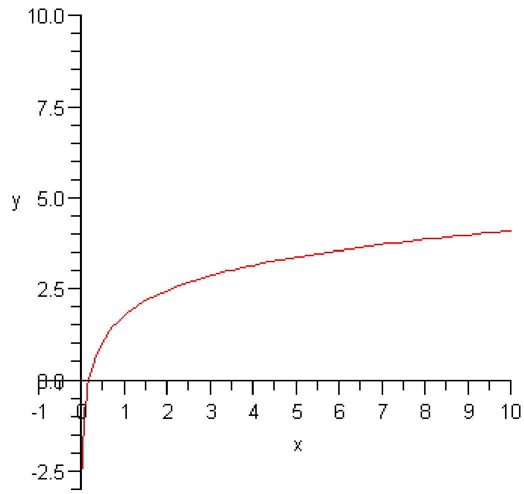
$$T(r) = 14.29\ln(0.0041r + 1)$$
$$40 = 14.29\ln(0.0041r + 1)$$
$$\frac{40}{14.29} = \frac{14.29\ln(0.041r + 1)}{14.29}$$
$$2.799 = \ln(0.0041r + 1)$$
$$\Rightarrow e^{2.799} = 0.0041r + 1$$
$$\Rightarrow 0.0041r = e^{2.799} - 1$$
$$\Rightarrow r = \frac{e^{2.799} - 1}{0.0041}$$
$$\Rightarrow r \approx 3763$$

Example 8

Graph $y = \log 6x$

X	y
2	$y = \log(6(2)) = \log(12) = 1.07$
10	$y = \log(6(10)) = \log(60) = 1.8$
20	$y = \log(6(20)) = \log(120) = 2.1$
40	$y = \log(6(40)) = \log(240) = 2.4$

Plot the given values from the table gives the following graph



Example 9

Graph $y = 5\log(x+1)$

x	Y
2	$y = 5\log(2+1) = 5\log(3) = 2.4$
10	$y = 5\log(10+1) = 5\log(11) = 5.2$
20	$y = 5\log(20+1) = 5\log(21) = 6.6$
40	$y = 5\log(40+1) = 5\log(41) = 8.1$

Plot the given values from the table gives the following graph