Math 151
Section 3.1

Derivatives of polynomials

Review of the limit definition of a derivative.

$f(x)$	$f'(x)$
$f(x)=4x+5$	$f'(x)=4$ **See Example 1 Section 2.7**
$f(x)=x^2-2x$	$f'(x)=2x-2$ **See Example 3 Section 2.7**
$f(x)=x^3+4$	$f'(x)=3x^2$ **See Example 4 Section 2.7**
$f(x)=2x^4-4x^3$	$f(x)=4x^3-12x^2$

Power Rule

If $f(x)=cx^n$, then $f(x)=ncx^{n-1}$

Constant Rule

If $f(x)=C$ where C is a constant, then $f(x)=0$

Proof of the power rule

Let $f(x)=x^n$, *and find* $f'(x)$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}$$

$$=\lim_{h\to 0}\frac{\left[x^n+nx^{n-1}h+\frac{n(n-1)}{2}x^{n-2}h^2+....+nxh^{n-1}+h^n\right]-x^n}{h}$$

$$=\lim_{h\to 0}\frac{nx^{n-1}h+\frac{n(n-1)}{2}xn^{n-2}h^2+......+h^n}{h}$$

$$=nx^{n-1}h+0+0+.....+0$$

$$=nx^{n-1}$$

Examples using the power rule.

Example 1

If $f(x) = 2x^3$, then find $f'(x)$ using the power rule.

$f(x) = 3 \cdot 2x^{3-1} = 6x^2$

Example 2

If $f(x) = x^4 + 2x^3 + 3x$, then find $f'(x)$ using the power rule.

$f'(x) = 4x^{4-1} + 3 \cdot 2x^{3-1} + 3 = 4x^3 + 6x^2 + 3$

Example 3

If $f(x) = \dfrac{1}{x^2}$, then find $f'(x)$ using the power rule.

$f(x) = \dfrac{1}{x^2} = x^{-2}$

$f(x) = -2x^{-2-1} = -2x^{-3} = \dfrac{-2}{x^3}$

Example 4

If $f(x) = \dfrac{1}{x^3} + \dfrac{4}{x^2}$, then find $f'(x)$ using the power rule.

$f(x) = \dfrac{1}{x^3} + \dfrac{4}{x^2} = x^{-3} + 4x^{-2}$

$f(x) = -3x^{-3-1} + (-2)4x^{-2-1} = -3x^{-4} - 8x^{-3} = -\dfrac{3}{x^4} - \dfrac{8}{x^3}$

Example 5

If $f(x) = \sqrt{x}$, then find $f'(x)$ using the power rule.

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}$$

$$f(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$$

Example 6

If $f(x) = 6\sqrt{x}$, then find $f'(x)$ using the power rule.

$$f(x) = 6\sqrt{x} = 6x^{\frac{1}{2}}$$

$$f(x) = \frac{1}{2}(6)x^{\frac{1}{2}-1} = 3x^{-\frac{1}{2}} = \frac{3}{x^{\frac{1}{2}}} = \frac{3}{\sqrt{x}}$$

Example 7

If $f(x) = 2\sqrt[3]{x}$, then find $f'(x)$ using the power rule.

$$f(x) = 2\sqrt[3]{x} = 2x^{\frac{1}{3}}$$

$$f(x) = \frac{1}{3}\cdot 2x^{\frac{1}{3}-1} = \frac{2}{3}x^{-\frac{2}{3}} = \frac{2}{3x^{\frac{2}{3}}} = \frac{2}{3\sqrt[3]{x^2}}$$

The exponential function

The definition of the number e

e is the number such that $\lim_{h\to 0}\frac{e^h - 1}{h} = 1$

The derivative of the exponential function

$f(x) = e^x$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{e^{x+h} - e^h}{h} = \lim_{h\to 0}\frac{e^x e^h - e^x}{h} = \lim_{h\to 0}\frac{e^x\left(e^h - 1\right)}{h}$$
$$= \lim_{h\to 0} e^x\left(\frac{e^h - 1}{h}\right) = e^x\left[\lim_{h\to 0}\left(\frac{e^h - 1}{h}\right)\right] = e^x(1) = e^x$$

Definition: The derivative of $f(x) = e^x$ is $f'(x) = e^x$

Example 8

Find the derivative of $f(x) = 6e^x$

$f'(x) = 6e^x$

Example 9

Find the equation of a tangent line to the function $f(x) = x^2 + x$ through the point (1,2).

Find the derivative of $f(x) = x^2 + x$

$f'(x) = 2x + 1$

Find the slope of the tangent line: $m = f'(1) = 2(1) + 1 = 2 + 1 = 3$

$$\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 2 &= 3(x - 1) \\
y - 2 &= 3x - 3 \\
y - 2 + 2 &= 3x - 3 + 2 \\
y &= 3x - 1
\end{aligned}$$

Example 10

Find the equation of a tangent line to the function $f(x) = x^3 + e^x$ through the point (0,1).

Find the derivative of $f(x) = x^3 + e^x$

$f'(x) = 3x^2 + e^x$

Find the slope of the tangent line: $m = f'(0) = 3(0)^2 + e^0 = 0 + 1 = 1$

$$y - y_1 = m(x - x_1)$$
$$y - 1 = 1(x - 0)$$
$$y - 1 = x$$
$$y - 1 + 1 = x + 1$$
$$y = x + 1$$

Example 11

The equation of motion of a particle is $s(t) = t^3 - 4t$ where s is in meters and t is seconds.

a) Find the velocity and acceleration as a function of t.

$$v(t) = s'(t) = 3t^{3-1} - 4 = 3t^2 - 4$$
$$a(t) = s''(t) = 6t$$

b) Find the velocity after 2 seconds.

$$v(t) = 3t^{3-1} - 4 = 3t^2 - 4$$
$$v(2) = 3(2)^2 - 4 = 12 - 4 = 8\ \frac{m}{s}$$

c) Find the acceleration after 2 seconds.

$$a(t) = s''(t) = 6t$$
$$a(2) = 6(2) = 12\ \frac{m}{s^2}$$