

## Math 151

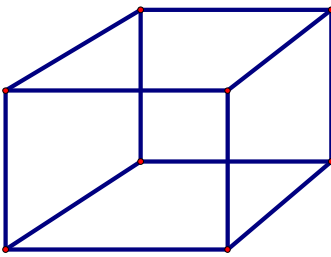
### Section 3.6

### Optimization

#### Example 1

A manufacturer wants to design an open box having a square base and a surface area of 108 square inches, as shown in Figure 1-1. What dimensions will produce a box with maximum volume?

Figure 1.1



The Volume of the open box would be:

$$V = x^2h$$

The surface area would be:

$$A = 4x \text{ (Area of each face)} + \text{Area of bottom}$$

$$A = 4xh + x \cdot x$$

$$A = 4xh + x^2$$

Since the surface area is 108 square inches, the new formula would be:

$$108 = 4xh + x^2$$

Solve this equation for h

$$108 = 4xh + x^2$$

$$4xh = 108 - x^2$$

$$h = \frac{108 - x^2}{4x}$$

Now, you can express the volume in terms of  $x$ .

$$V = x^2 h = x^2 \left( \frac{108 - x^2}{4x} \right) = \frac{1}{4} (108x - x^3)$$

You can maximize the volume by taking the derivative.

$$V = \frac{1}{4} (108x - x^3) = 27x - \frac{1}{4} x^3$$

$$V' = 27 - \frac{3}{4} x^2$$

Next, set the derivative equal to zero.

$$27 - \frac{3}{4} x^2 = 0$$

$$-\frac{3}{4} x^2 = -27$$

$$x^2 = \frac{4}{3} \cdot 27$$

$$x^2 = 36$$

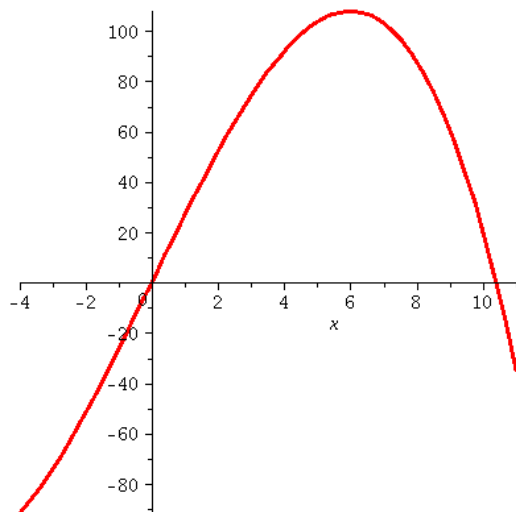
$$\sqrt{x^2} = \sqrt{36}$$

$$x = \pm 6$$

Since we cannot have a negative distance, the only length of the box is 6 inches. The height

would be as follows:  $h = \frac{108 - x^2}{4x} = \frac{108 - (6)^2}{4(6)} = \frac{108 - 36}{24} = \frac{72}{24} = 3 \text{ in}$ . The dimensions of the

box would be 6 inches by 6 inches by 3 inches. Here is the graph of the volume as function of  $x$ .



## Example 2

Find the two positive numbers that satisfy the given requirements. The product 196 and the sum is a minimum.

Let  $x$  be the first number and  $y$  be the second number.

$$xy = 196$$

$$S = x + y$$

First solve the first equation for  $y$ .

$$xy = 196$$

$$\frac{xy}{x} = \frac{196}{x}$$

$$y = \frac{196}{x}$$

Now, take the derivative of  $S$

$$S = x + \frac{196}{x}$$

$$S = x + 196x^{-1}$$

$$S' = 1 - 196x^{-2}$$

$$S' = 1 - \frac{196}{x^2}$$

$$0 = 1 - \frac{196}{x^2}$$

$$\frac{196}{x^2} = 1$$

$$\frac{196}{x^2} x^2 = 1 \cdot x^2$$

$$196 = x^2$$

$$\sqrt{196} = \sqrt{x^2}$$

$$x = \sqrt{196}$$

$$x = 14$$

The value of y would be

$$y = \frac{196}{14} = 14$$

Therefore, the two numbers would be 14 and 14.

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### Example 3

Suppose that you have 320 square centimeters of aluminum to make a cylinder shaped coke can. Find the radius of the can that would produce a maximum volume.

First find formula for the volume of the coke can.

$$\text{Surface area of cylinder: } S = 2\pi r + 2\pi r h$$

$$\text{Volume of a cylinder: } V = \pi r^2 h$$

**First, take the surface area formula and solve for h**

$$S = 2\pi r + 2\pi r h$$

$$320 = 2\pi r + 2\pi r h$$

$$320 - 2\pi r = 2\pi r h$$

$$\Rightarrow h = \frac{320 - 2\pi r}{2\pi r}$$

**Next, substitute  $\frac{320 - 2\pi r}{2\pi r}$  in for h in the volume formula.**

$$V = \pi r^2 h$$

$$V = \pi r^2 \left( \frac{320 - 2\pi r}{2\pi r} \right)$$

$$V = \frac{320\pi r}{2\pi r} - \frac{2\pi r^3}{2\pi r}$$

$$V = 160r - \pi r^3$$

Now, take the derivative of the volume and set the result equal to zero

$$V' = 160 - 3\pi r^2$$

$$0 = 160 - 3\pi r^2$$

$$3\pi r^2 = 160$$

$$r^2 = \frac{160}{3\pi}$$

$$r^2 = 16.98 \Rightarrow r = \sqrt{16.98} \Rightarrow r = \pm 4.1 \text{ cm}$$

Eliminating the negative answer we get the radius of the can is 4.1 cm.

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#### Example 4

Find the length and width of a rectangle that has a perimeter of 120 meters and a maximum area.

Key formulas:

$$A = lw$$

$$P = 2l + 2w$$

The perimeter would be  $120 = 2w + 2l$

Now, solve equation for w.

$$120 = 2w + 2l$$

$$2w = 120 - 2l$$

$$\frac{2w}{2} = \frac{120 - 2l}{2}$$

$$w = 60 - l$$

Express Area as a function of w.

$$A = lw = (60 - l)l = 60l - l^2$$

$$A' = 60 - 2l$$

Now, set the derivative of the area equal to zero.

$$A' = 60 - 2l$$

$$0 = 60 - 2l$$

$$2l = 60$$

$$l = 30$$

Now, solve for w

$$w = 60 - l = 60 - 30 = 30$$

Therefore, the rectangle would be 30 meters by 30 meters.

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#### Example 4

Find the length and width of a rectangle that has an area of 100 square meters and a maximum perimeter.

Area formula:  $A = l \cdot w$

$$100 = l \cdot w$$

$$\frac{100}{l} = \frac{lw}{l}$$

$$w = \frac{100}{l}$$

The perimeter of a rectangle is  $P = 2l + 2w$

$$P = 2l + 2\left(\frac{100}{l}\right)$$

$$P = 2l + \frac{200}{l}$$

### Find the derivative of P

$$P = 2l + 200l^{-1}$$

$$P' = 2 - 200l^{-2}$$

$$2 - 200l^{-2} = 0$$

$$\frac{200}{l^2} = 2$$

$$2l^2 = 200$$

$$l^2 = 100$$

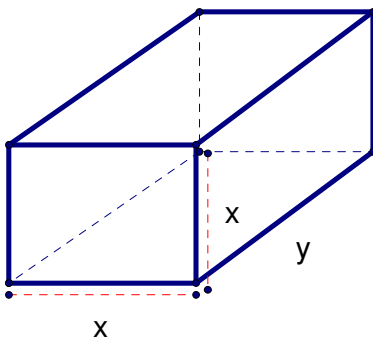
$$l = \sqrt{100}$$

$$l = 10 \text{ m}^2$$

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### Example 6

A rectangle package to be sent by a postal service can have a maximum combined length and girth (perimeter of a cross section) of 108 inches. Find the dimensions of the package of maximum volume that can be sent.



### Key Formulas:

$$\text{Girth} = 2x + 2x + y = 4x + y$$

$$108 = 4x + y$$

$$V = lwh = x \cdot y \cdot x = x^2y$$

Expressed the volume in terms of x

$$108 = 4x + y$$

$$\Rightarrow y = 108 - 4x$$

$$V = x^2y$$

$$V = x^2(108 - 4x)$$

$$V = 108x^2 - 4x^3$$

Find the derivative of V

$$V' = 216x - 12x^2$$

$$216x - 12x^2 = 0$$

$$12x(18 - x) = 0$$

$$12x = 0 \quad \text{or} \quad 18 - x = 0$$

$$x = 0 \text{ in} \quad \text{or} \quad x = 18 \text{ in} \Rightarrow \text{length: } x = 18 \text{ in}$$

$$y = 108 - 4x$$

$$y = 108 - 4(18)$$

$$y = 108 - 72$$

$$y = 36 \text{ in}$$

The dimensions that would give a maximum volume would be 18 inches by 18 inches by 36 inches.