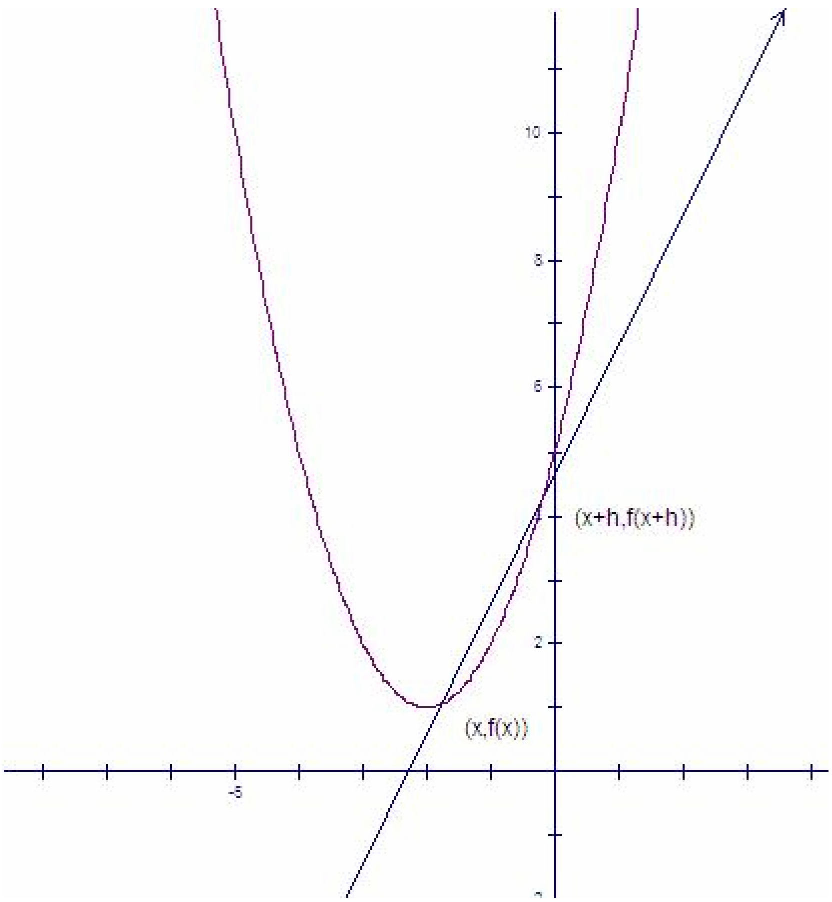
Math 151

Section 2.7

Limit Definition of a Derivative

Review of Slope

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



$$m = \lim_{h \to 0} \frac{y_2 - y_1}{x_2 - x_1} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Limit Definition of a Derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example 1

Find the derivative of $f(x) = 4x - 5$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{4(x+h)+5-(4x+5)}{h} = \lim_{h\to 0}\frac{4x+4h+5-4x-5}{h} = \lim_{h\to 0}\frac{4h}{h} = \lim_{h\to 0}4 = 4$$

Example 2

Find the derivative of $f(x) = x^2 + 4$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2 + 4 - (x^2 - 4)}{h} = \lim_{h\to 0}\frac{(x+h)(x+h) + 4 - (x^2 - 4)}{h}$$

$$= \lim_{h\to 0}\frac{x^2 + xh + xh + h^2 + 4 - x^2 + 4}{h} = \lim_{h\to 0}\frac{2xh + h^2}{h} = \lim_{x\to 0}\frac{h(2x+h)}{h} = \lim_{h\to 0}2x + h = 2x + 0 = 2x$$

Example 3

Find the derivative of $f(x) = x^2 - 2x$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2 - 2(x+h) - (x^2 - 2x)}{h} = \lim_{h\to 0}\frac{x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x}{h}$$

$$= \lim_{h\to 0}\frac{h^2 + 2xh - 2h}{h} = \lim_{h\to 0}\frac{h(h + 2x - 2)}{h} = \lim_{h\to 0}h + 2x - 2 = 2x - 2$$

Example 4

Find the derivative of $f(x) = x^3 + 4$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^3 + 4 - (x^3 + 4)}{h} = \lim_{h\to 0}\frac{x^3 + 3x^2h + 3xh^2 + h^2 + 4 - x^3 - 4}{h}$$

$$= \lim_{h\to 0}\frac{3x^2h + 3xh^2 + h^h}{h} = \lim_{h\to o}\frac{h(3x^2 + 3xh + h)}{h} = \lim_{h\to 0}3x^2 + 3xh + h = 3x^2 + 3x(0) + 0 = 3x^2$$

Example 5

Find the derivative of $f(x)=\dfrac{1}{x+2}$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{1}{(x+h)+2}-\frac{1}{x+2}}{h}=\lim_{h\to 0}\frac{\frac{1}{x+h+2}-\frac{1}{x+2}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{x+2}{(x+h+2)(x+2)}-\frac{x+h+2}{(x+h+2)(x+2)}}{h}=\lim_{h\to 0}\frac{\frac{x+2-(x+h+2)}{(x+h+2)(x+2)}}{h}=\lim_{h\to 0}\frac{\frac{x+2-x-h-2}{(x+h+2)(x+2)}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{-h}{(x+h+2)(x+2)}}{h}=\lim_{h\to 0}\frac{\frac{-h}{(x+h+2)(x+2)}\cdot\frac{1}{h}}{h\cdot\frac{1}{h}}=\lim_{h\to 0}\frac{-1}{(x+h+2)(x+2)}=\frac{-1}{(x+0+2)(x+2)}$$

$$=\frac{-1}{(x+2)^2}\ or\ \frac{1}{x^2+4x+4}$$

Example 6

Find the derivative of the function $f(x)=\sqrt{x-3}$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{x+h-3}-\sqrt{x-3}}{h}=\lim_{h\to 0}\frac{\left(\sqrt{x+h-3}-\sqrt{x-3}\right)}{h}\frac{\left(\sqrt{x+h-3}+\sqrt{x-3}\right)}{\left(\sqrt{x+h-3}-\sqrt{x-3}\right)}$$

$$=\lim_{h\to 0}\frac{\left(\sqrt{x+h-3}\right)^2-\sqrt{x+h-3}\sqrt{x-3}+\sqrt{x+h-3}\sqrt{x-3}-\left(\sqrt{x-3}\right)^2}{h\left(\sqrt{x+h-3}-\sqrt{x-3}\right)}=\lim_{h-o}\frac{x+h-3-(x-3)}{h\left(\sqrt{x+h-3}+\sqrt{x-3}\right)}$$

$$=\lim_{h\to 0}\frac{x+h-3-x+3}{h\left(\sqrt{x+h-3}+\sqrt{x-3}\right)}=\lim_{h\to 0}\frac{h}{h\left(\sqrt{x+h-3}+\sqrt{x-3}\right)}=\lim_{h\to 0}\frac{1}{\sqrt{x+h-3}+\sqrt{x-3}}=\frac{1}{\sqrt{x+0-3}+\sqrt{x-3}}$$

$$=\frac{1}{\sqrt{x-3}+\sqrt{x-3}}=\frac{1}{2\sqrt{x-3}}$$

Example 7

Find the derivative of $f(x) = -2x + 3$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{-2(x+h)+3-(-2x+3)}{h} = \lim_{h\to 0}\frac{-2x-2h+3+2x-3}{h} = \lim_{h\to 0}\frac{-2h}{h} = \lim_{h\to 0} -2 =$$

Example 8

Find the slope and the equation of a tangent to the function $f(x) = x^2 - 2$ at the point $(1,-1)$.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2 - 2 - (x^2 - 2)}{h} = \lim_{h\to 0}\frac{(x+h)(x+h) - 2 - (x^2 - 2)}{h}$$

$$= \lim_{h\to 0}\frac{x^2 + xh + xh + h^2 - 2 - x^2 + 2}{h} = \lim_{h\to 0}\frac{2xh + h^2}{h} = \lim_{x\to 0}\frac{h(2x+h)}{h} = \lim_{h\to 0} 2x + h = 2x + 0 = 2x$$

$$f'(x) = 2x$$

$$m = f'(1) = 2(1) = 2$$

$$y - (-1) = 2(x - 1)$$
$$y + 1 = 2x - 2$$
$$y = 2x - 3$$

Example 9

Find the equation of tangent line to the function $f(x)=\dfrac{1}{x}$ at the point $\left(2,\dfrac{1}{2}\right)$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\dfrac{1}{(x+h)}-\dfrac{1}{x}}{h}=\lim_{h\to 0}\frac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h}$$

$$=\lim_{h\to 0}\frac{\dfrac{x}{(x+h)x}-\dfrac{x+h}{x(x+h)}}{h}=\lim_{h\to 0}\frac{\dfrac{x-(x+h)}{(x+h)x}}{h}=\lim_{h\to 0}\frac{\dfrac{x-x-h}{(x+h)x}}{h}$$

$$=\lim_{h\to 0}\frac{\dfrac{-h}{(x+h)x}}{h}=\lim_{h\to 0}\frac{\dfrac{-h}{(x+h)x}\cdot\dfrac{1}{h}}{h\cdot\dfrac{1}{h}}=\lim_{h\to 0}\frac{-1}{(x+h)x}=\frac{-1}{(x+0)x}=-\frac{1}{x^2}$$

$$m=f'(2)=\frac{1}{2}$$

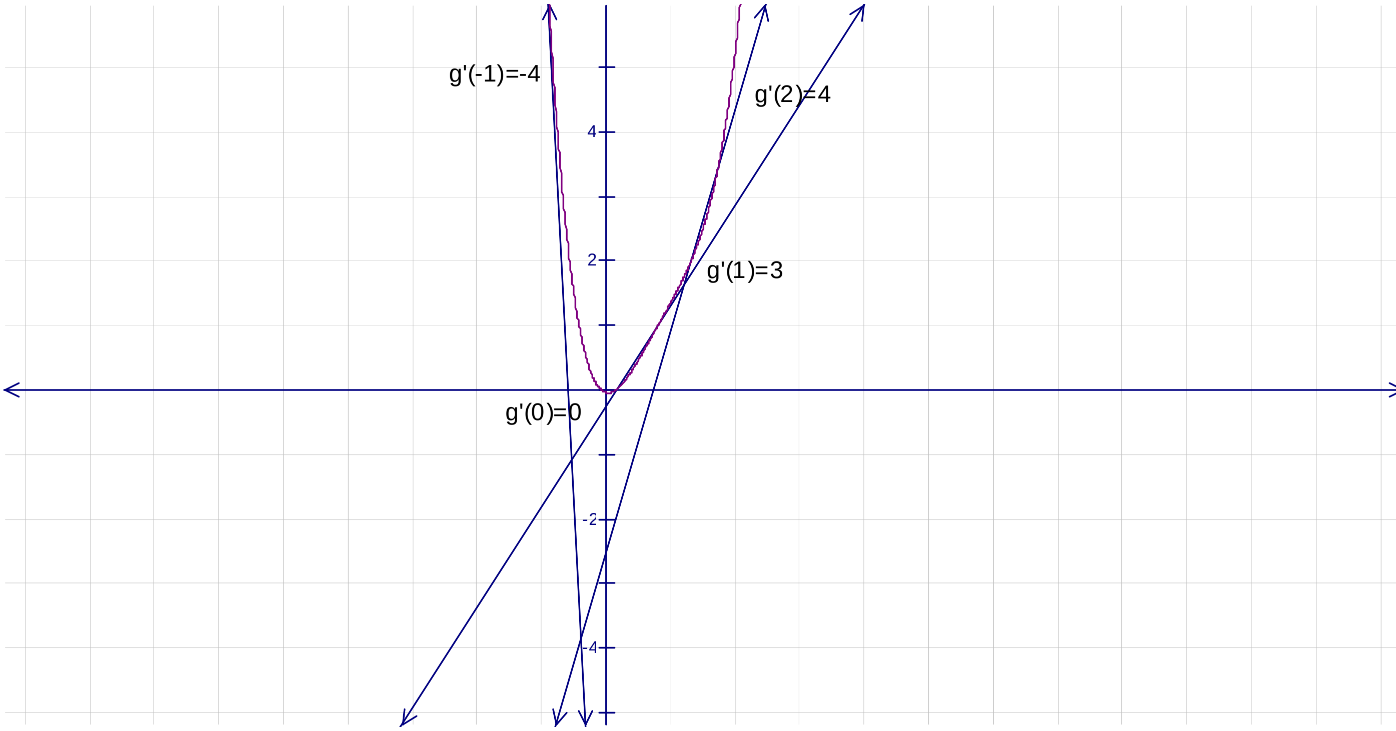
$$y-\frac{1}{2}=\frac{1}{2}(x-2)$$

$$y-\frac{1}{2}=\frac{1}{2}x-1$$

$$y=\frac{1}{2}x-\frac{1}{2}$$

Example 10

Sketch a graph of a function for which
$g(0) = 0,\ g'(0) = 0,\ g'(-1) = -4,\ g'(1) = 3,\ g'(2) = 4$



Example 11

A particle that moves along a straight line path is given by the formula $f(t) = 60t - 4.9t^2$.
Find the velocity of the object when $t = 5$

$$f'(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} = \lim_{h\to 0} \frac{60(t+h) - 4.9(t+h)^2 - \left(60t - 4.9t^2\right)}{h}$$

$$= \lim_{h\to 0} \frac{60t + 60h - 4.9\left(t^2 + 2th + h^2\right) - 60t + 4.9t^2}{h} = \lim_{h\to 0} \frac{60t + 60h - 4.9t^2 - 9.8th - 4.9h^2 - 60t + 4.9t^2}{h}$$

$$= \lim_{h\to 0} \frac{60h - 9.8th - 4.9h^2}{h} = \lim_{h\to 0} \frac{h\left(60 - 9.8t - 4.9h\right)}{h} = \lim_{h\to 0} 60 - 9.8t - 4.9h = 60 - 9.8t$$

Find $f'(t)$ at $x = 5$

$$f'(5) = 60 - 9.8(5) = 60 - 49 = 11\ m/s$$