

Math 151
Section 2.5

Implicit Differentiation

The functions we have use so far have been described in terms of one variable.

$$y = x^2 + 4x$$

$$y = \sin(x)$$

Functions can also be described implicitly, in other words described in terms of two or more variables.

$$x^2 + y^2 = 4$$

$$2x^2 + y^2 + 5x = 6$$

This is how to find the derivative of a function using implicit differentiation.

Example 1

Find the derivative using implicit differentiation.

$$x^2 + 2y^2 = 3$$

$$x^2 + 2y^2 = 3$$

$$\frac{d}{dx}(x^2 + 2y^2) = \frac{d}{dx} 3$$

$$\frac{d}{dx} x^2 + \frac{d}{dx} 2y^2 = 0$$

$$2x + 4y \frac{dy}{dx} = 0$$

$$4y \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{2x}{4y}$$

$$\frac{dy}{dx} = -\frac{x}{2y}$$

Example 2

Find the derivative using implicit differentiation.

$$x^2 + xy + 6y^2 = 5$$

$$x^2 + xy + 6y^2 = 5$$

$$\frac{d}{dx}(x^2 + xy + 6y^2) = \frac{d}{dx}(5)$$

$$\frac{d}{dx}x^2 + \frac{d}{dx}xy + \frac{d}{dx}6y^2 = 0$$

$$2x + y + x\frac{dy}{dx} + 12y\frac{dy}{dx} = 0$$

$$x\frac{dy}{dx} + 12y\frac{dy}{dx} = -2x - y$$

$$(x + 12y)\frac{dy}{dx} = -(2x + y)$$

$$\frac{dy}{dx} = -\frac{2x + y}{x + 12y}$$

Example 3

Find the derivative using implicit differentiation $x^2y + xy^2 = 9x^2$

$$x^2y + xy^2 = 9x^2$$

$$\frac{d}{dx}(x^2y + xy^2) = \frac{d}{dx}(9x^2)$$

$$2xy + x^2y' + y^2 + 2xyy' = 18x$$

$$x^2y' + 2xyy' = 18x - 2xy - y^2$$

$$(x^2 + 2xy)y' = 18x - 2xy - y^2$$

$$y' = \frac{18x - 2xy - y^2}{x^2 + 2xy}$$

Example 4

Find the derivative using implicit differentiation $x^2 y^2 + x^2 \cos y = 6$

$$x^2 y^2 + x^2 \cos y = 6$$

$$\frac{d}{dx}(x^2 y^2 + x^2 \cos y) = \frac{d}{dx} 6$$

$$2xy^2 + 2x^2 yy' + 2x \cos y - x^2 \sin y y' = 0$$

$$(2x^2 y + x^2 \cos y)y' + 2xy^2 + 2x \cos y = 0$$

$$(2x^2 y + x^2 \cos y)y' = -(2xy^2 + 2x \cos y)$$

$$y' = -\frac{2x^2 y + 2x \cos y}{2x^2 y + x^2 \cos y}$$

$$y' = \frac{2xy + 2 \cos y}{2xy + x \cos y}$$

Example 5

Find the derivative using implicit differentiation $y^2 e^x + 5x = 6y$

$$y^2 e^x + 5x = 6y$$

$$\frac{d}{dx}(y^2 e^x + 5x) = \frac{d}{dx} 6y$$

$$y^2 e^x + 2ye^x y' + 5 = 6y'$$

$$2ye^x y' - 6y' = -y^2 e^x - 5$$

$$(2ye^x - 6)y' = -(y^2 e^x + 5)$$

$$y' = -\frac{y^2 e^x + 5}{2ye^x - 6}$$

Example 6

Use implicit differentiation to find the equation of a tangent line to the curve at the given point. $2x^2 + 3xy + y^2 = 6$; (1,1)

$$2x^2 + 3xy + y^2 = 6$$

$$\frac{d}{dx}(2x^2 + 3xy + y^2) = \frac{d}{dx}(6)$$

$$4x + 3y + 3xy' + 2yy' = 0$$

$$3xy' + 2yy' = -4x - 3y$$

$$(3x + 2y)y' = -(4x + 3y)$$

$$y' = -\frac{4x + 3y}{3x + 2y}$$

$$m = -\frac{4(1) + 3(1)}{3(1) + 2(1)} = -\frac{4 + 3}{3 + 2} = -\frac{7}{5}$$

$$y - 1 = -\frac{7}{5}(x - 1)$$

$$y - 1 = -\frac{7}{5}x + \frac{7}{5}$$

$$y = -\frac{7}{5}x + \frac{12}{5}$$

Example 7

Use implicit differentiation to find the equation of a tangent line to the curve at the given point. $x^2 - xy + y^2 = 1$; (1,1)

$$x^2 - xy + y^2 = 1$$

$$\frac{d}{dx}(x^2 - xy + y^2) = \frac{d}{dx}(1)$$

$$2x - y - xy' + 2yy' = 0$$

$$-xy' + 2yy' = -2x + y$$

$$(2y - x)y' = y - 2x$$

$$y' = \frac{y - 2x}{2y - x}$$

$$m = \frac{1 - 2(1)}{2(1) - 1} = \frac{-1}{1} = -1$$

$$y - 1 = -1(x - 1)$$

$$y - 1 = -x + 1$$

$$y = -x + 2$$

Example 8

Find the derivative using implicit differentiation $2 \sin x \cos y = 1$

$$2 \sin x \cos y = 1$$

$$\frac{d}{dx}(2 \sin x \cos y) = \frac{d}{dx}(1)$$

$$2 \cos x \cos y - 2 \sin x \sin y y' = 0$$

$$-2 \sin x \sin y y' = -2 \cos x \cos y$$

$$y' = \frac{-2 \cos x \cos y}{-2 \sin x \sin y}$$

$$y' = \cot x \cot y$$

Example 9

Find the derivative $\frac{dy}{dx}$ using implicit differentiation.

$$\sin x + 4 \cos 3y = 1$$

Find the derivative

$$\frac{d}{dx}(\sin x + 4 \cos 3y) = \frac{d}{dx}(1)$$

$$\cos x - 4 \sin 3y(3) \frac{dy}{dx} = 0$$

$$\cos x - 12 \sin 3y \frac{dy}{dx} = 0$$

$$-12 \sin 3y \frac{dy}{dx} = -\cos x$$

$$\frac{dy}{dx} = \frac{-\cos x}{-12 \sin 3y}$$

$$\frac{dy}{dx} = \frac{\cos x}{12 \sin 3y}$$

Example 10

Find the derivative by implicit differentiation and find the slope of the tangent to the curve at the given point.

$$x^3 - y^3 = 0: (1,1)$$

$$\frac{d}{dx}(x^3 - y^3) = \frac{d}{dx}(0)$$

$$3x^2 - 3y^2 \frac{dy}{dx} = 0$$

$$-3y^2 \frac{dy}{dx} = -3x^2$$

$$\frac{dy}{dx} = \frac{-3x^2}{-3y^2} \Rightarrow \frac{dy}{dx} = \frac{y^2}{x^2} \Rightarrow m = \frac{1^2}{1^2} = 1$$

Example 11

Find the derivative using implicit differentiation, and evaluate the derivative at the indicated point

$$xy^2 = 6: (6,1)$$

$$\frac{d}{dx}(xy^2) = \frac{d}{dx}(6)$$

$$y^2 + x(2y)\frac{dy}{dx} = 0$$

$$y^2 + 2xy\frac{dy}{dx} = 0$$

$$2xy\frac{dy}{dx} = -y^2$$

$$\frac{dy}{dx} = -\frac{y^2}{2xy}$$

$$\frac{dy}{dx} = -\frac{y}{2x}$$

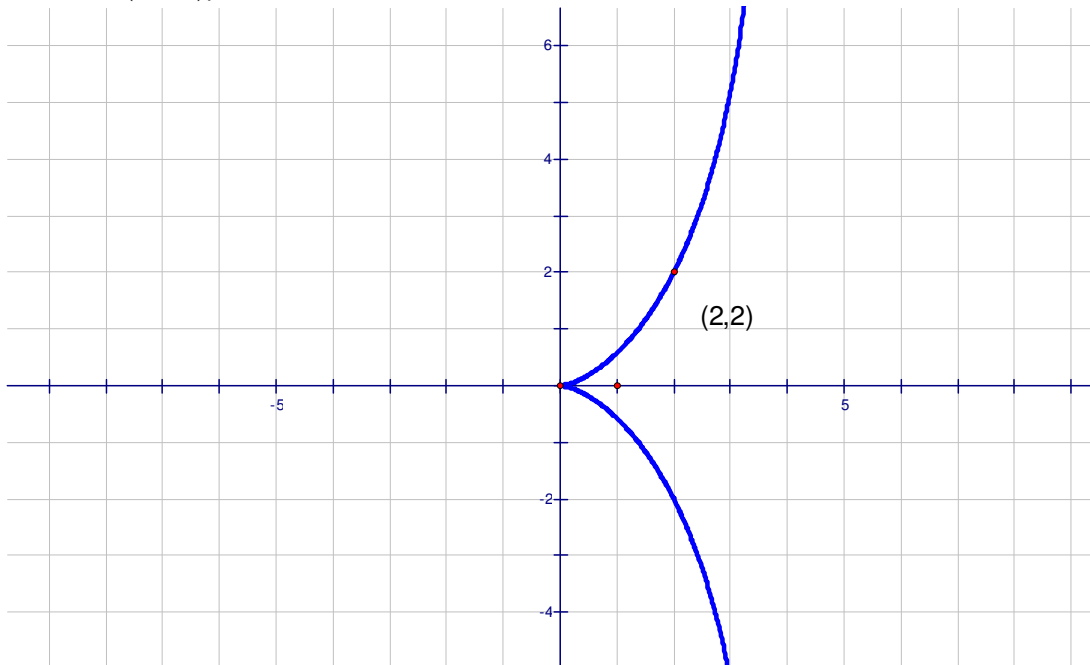
$$\frac{dy}{dx} = -\frac{1}{2(6)} = -\frac{1}{12}$$

Example 12

Famous Curves

Find the slope of the tangent line to the graph at the point (2,2)

Cissoid: $(4-x)y^2 = x^3$



$$(4-x)y^2 = x^3$$

$$\frac{d}{dx} [(4-x)y^2] = \frac{d}{dx} x^3$$

$$\frac{d}{dx} (4-x)y^2 + \frac{d}{dx} y^2(4-x) = 3x^2$$

$$(-1)y^2 + 2yy'(4-x) = 3x^2$$

$$-y^2 + 2y(4-x)y' = 3x^2$$

$$(8y - 2xy)y' = y^2 + 3x^2$$

$$y' = \frac{y^2 + 3x^2}{8y - 2xy}$$

$$\text{At } (2,2): y' = \frac{y^2 + 3x^2}{8y - 2xy} = \frac{2^2 + 3 \cdot 2^2}{8(2) - 2(2)(2)} = \frac{4 + 12}{16 - 8} = \frac{16}{8} = 2$$